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Homework grading scheme

Each problem is worth ten points. Points for a problem are assessed as follows:
Hints

Assignment 2

1.43. If \( f = \sum_{i=0}^{\infty} f_i, g = \sum_{i=0}^{\infty} g_i \in R[[X_1, \ldots, X_n]] \) then \( fg = \sum_{i=0}^{\infty}(\sum_{j=0}^{i} f_j g_{i-j}) \) (see Sharp p. 11). Hence, \( fg = 1 \) if and only if

\[
\begin{align*}
1 &= f_0 g_0, \\
0 &= f_0 g_1 + f_1 g_0, \\
0 &= f_0 g_2 + f_1 g_1 + f_2 g_0, \\
&\vdots
\end{align*}
\]

2.5 Use the binomial theorem, which is valid in any commutative ring \( R \): If \( x, y \in R \), and \( n \in \mathbb{N} \), then

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

2.22 It may be useful to use the (total) degree function \( \deg : K[X_1, X_2] - 0 \to \mathbb{N} \) (see p. 9 of Sharp). This function satisfies \( \deg(pq) = \deg(p) + \deg(q) \) for non-zero elements \( p \) and \( q \) of \( K[X_1, X_2] \).

Assignment 5

For Exercise 3.50 and Exercise 3.51 consider using Corollary 3.49. For Exercise 4.7 first read and understand Exercise 2.46. For Exercise 4.8 consider using Exercise 4.7.

Assignment 6

For Exercise 4.28, prove that the ideal \((X^3, XY, Y^n)\) is primary by finding a maximal ideal \( M \) and \( k \in \mathbb{N} \) such that

\[
M^k \subseteq (X^3, XY, Y^n) \subseteq M,
\]

take radicals, and apply Proposition 4.9.

Assignment 8

For Exercise 5.34, assume that \( R \) does admit a non-zero nilpotent element \( x \) and obtain a contradiction via the following idea. Consider \( I = \{ r \in R : rx = 0 \} \). Then \( I = (0 : x) \), and \( I \) is thus
an ideal of $R$. If $I = R$, then $1 \cdot x = 0$, which is a contradiction. Assume that $I \subsetneq R$, so that $I$ is a proper ideal. Since $I$ is a proper ideal, $I$ is included inside a maximal ideal $M$. Since $M$ is a maximal ideal, $M$ is a prime ideal. Consider $R_M$ and the image $x/1$ in $R_M$ of $x$ under the natural map. The element $x/1$ is nilpotent. By the hypothesis of this exercise we must have $x/1 = 0/1$. Now obtain the final contradiction.

**Suggested solutions to selected problems**

**Assignment 1**

1.16 Let $R'$ be a commutative ring, and let $\xi_1, \ldots, \xi_n \in R'$ be algebraically independent over the subring $R$ of $R'$. Let $T$ be a commutative $R$-algebra with structural ring homomorphism $f : R \rightarrow T$ and let $\alpha_1, \ldots, \alpha_n \in T$. Show that there is exactly one ring homomorphism

$$g : R[\xi_1, \ldots, x_n] \rightarrow T$$

which extends $f$ (that is, is such that $g|_R = f$) and is such that $g(\xi_i) = \alpha_i$ for all $i = 1, \ldots, n$.

**Suggest solution:** We begin with some notation. For $\lambda = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$ we will write

$$\xi^\lambda = \xi_{i_1}^{i_1} \cdots \xi_{i_n}^{i_n}.$$

With this notation every element $p$ of $R[\xi_1, \ldots, \xi_n]$ can be written uniquely in the form

$$p = \sum_{\lambda \in \mathbb{N}_0^n} r_\lambda \xi^\lambda$$

where $r_\lambda \in R$ for $\lambda \in \mathbb{N}_0^n$ and $r_\lambda = 0$ for all but finitely many $\lambda \in \mathbb{N}_0^n$ (see 1.14). If

$$q = \sum_{\lambda \in \mathbb{N}_0^n} s_\lambda \xi^\lambda$$

is another element of $R[\xi_1, \ldots, \xi_n]$, then we have

$$p + q = \sum_{\lambda \in \mathbb{N}_0^n} (r_\lambda + s_\lambda) \xi^\lambda,$$

$$pg = \sum_{\lambda \in \mathbb{N}_0^n} \left( \sum_{\begin{subarray}{c} \lambda_1, \lambda_2 \in \mathbb{N}_0^n, \\ \lambda_1 + \lambda_2 = \lambda \end{subarray}} r_{\lambda_1} s_{\lambda_2} \right) \xi^\lambda.$$
by

\[ g(p) = \sum_{\lambda \in \mathbb{N}_0^n} f(r_{\lambda}) \alpha^\lambda \]

for \( p \) as above; here, for \( \lambda = (i_1, \ldots, i_n) \in \mathbb{N}_0^n \) we define \( \alpha^\lambda = \alpha_1^{i_1} \cdots \alpha_n^{i_n} \). With \( p \) and \( q \) as above, and using that \( f \) is a ring homomorphism, we have:

\[ g(p + q) = g\left( \sum_{\lambda \in \mathbb{N}_0^n} (r_{\lambda} + s_{\lambda}) \xi^\lambda \right) = \sum_{\lambda \in \mathbb{N}_0^n} f(r_{\lambda} + s_{\lambda}) \alpha^\lambda = \sum_{\lambda \in \mathbb{N}_0^n} f(r_{\lambda}) \alpha^\lambda + \sum_{\lambda \in \mathbb{N}_0^n} f(s_{\lambda}) \alpha^\lambda = g(p) + g(q). \]

And:

\[ g(pq) = g\left( \sum_{\lambda \in \mathbb{N}_0^n} \left( \sum_{\lambda_1, \lambda_2 \in \mathbb{N}_0^n, \lambda_1 + \lambda_2 = \lambda} r_{\lambda_1} s_{\lambda_2} \right) \xi^\lambda \right) = \sum_{\lambda \in \mathbb{N}_0^n} f\left( \sum_{\lambda_1, \lambda_2 \in \mathbb{N}_0^n, \lambda_1 + \lambda_2 = \lambda} r_{\lambda_1} s_{\lambda_2} \right) \alpha^\lambda = \left( \sum_{\lambda \in \mathbb{N}_0^n} f(r_{\lambda}) \alpha^\lambda \right) \left( \sum_{\lambda \in \mathbb{N}_0^n} f(s_{\lambda}) \alpha^\lambda \right) = g(p)g(q). \]

It is clear that \( g(1) = 1 \). It follows that \( g \) is a ring homomorphism. It is also clear that \( g \) extends \( f \). Finally, to prove that \( g \) has the required uniqueness property, assume that \( h : R[\xi_1, \ldots, \xi_n] \to T \) is another right homomorphism such that \( h|_R = f \) and \( h(\xi_i) = \alpha_i \) for all \( i = 1, \ldots, n \). Let \( p \) be as above. We then have

\[ h(p) = h\left( \sum_{\lambda \in \mathbb{N}_0^n} r_{\lambda} \xi^\lambda \right) = \sum_{\lambda \in \mathbb{N}_0^n} h(r_{\lambda}) h(\xi^\lambda) = \sum_{\lambda \in \mathbb{N}_0^n} f(r_{\lambda}) \alpha^\lambda = g(p). \]
It follows that $h = g$.

1.19 Let $K$ be an infinite field, let $\Lambda$ be a finite subset of $K$, and let $f \in K[X_1, \ldots, X_n]$, the ring of polynomials over $K$ in the indeterminates $X_1, \ldots, X_n$. Suppose that $f \neq 0$. Show that there exist infinitely many choices of $(\alpha_1, \ldots, \alpha_n) \in (K - \Lambda)^n$ for which $f(\alpha_1, \ldots, \alpha_n) \neq 0$.

**Suggest solution:** We prove this by induction on $n$. The case $n = 1$ is clear because a non-zero polynomial in one variable over $K$ has finitely many distinct roots and $K - \Lambda$ is infinite. Assume that $n > 1$ and that the statement holds for $n - 1$; we will prove that it holds for $n$. There exists a non-negative integer $N$ such that $f(X_1, \ldots, X_n) = \sum_{k=0}^{N} f_k(X_1, \ldots, X_{n-1}) X_n^k$ where $f_k(X_1, \ldots, X_{n-1}) \in K[X_1, \ldots, X_{n-1}]$ for $k = 1, \ldots, N$, and $f_N(X_1, \ldots, X_{n-1})$ is non-zero. By the induction hypothesis, there exists $(\alpha_1, \ldots, \alpha_{n-1}) \in (K - \Lambda)^{n-1}$ such that $f_N(\alpha_1, \ldots, \alpha_{n-1}) \neq 0$. Consider the polynomial

$$g(X_n) = f(\alpha_1, \ldots, \alpha_{n-1}, X_n) = \sum_{k=0}^{N} f_k(\alpha_1, \ldots, \alpha_{n-1}) X_n^k$$

in the variable $X_n$. This polynomial is non-zero because $f_N(\alpha_1, \ldots, \alpha_{n-1}) \neq 0$. By the case $n = 1$, there exist infinitely many $\alpha_n \in K - \Lambda$ such that $g(\alpha_n) \neq 0$, i.e., $f(\alpha_1, \ldots, \alpha_n) \neq 0$; moreover, for any such $\alpha_n$ we have $(\alpha_1, \ldots, \alpha_n) \in (K - \Lambda)^n$. This proves the statement for $n$.

Assignment 2

1.43 Let $R$ be a commutative ring, and consider the ring $R[[X_1, \ldots, X_n]]$ of formal power series over $R$ in indeterminates $X_1, \ldots, X_n$. Let

$$f = \sum_{i=0}^{\infty} f_i \in R[[X_1, \ldots, X_n]],$$

where $f_i$ is either zero or a homogeneous polynomial of degree $i$ in $R[X_1, \ldots, X_n]$ (for each $i \in \mathbb{N}_0$). Prove that $f$ is a unit of $R[[X_1, \ldots, X_n]]$ if and only if $f_0$ is a unit of $R$.

**Suggest solution:** Assume that $f$ is a unit. Let $g \in R[[X_1, \ldots, X_n]]$ be such that $fg = 1$. Let $g = \sum_{i=0}^{\infty} g_i$ be the standard representation of $g$. Now

$$fg = \sum_{i=0}^{\infty} \left( \sum_{j=0}^{i} f_j g_{i-j} \right)$$


and this expression is the standard representation of \( fg \) in \( R[[X_1, \ldots, X_n]] \). Since \( fg = 1 \) we must therefore have

\[
1 = f_0g_0 \quad \text{and} \quad 0 = \sum_{j=0}^{i} f_jg_{i-j} \quad \text{for } i > 0.
\]

In particular, we see that \( f_0g_0 = 1 \), i.e., \( f_0 \) is a unit. Now assume that \( f_0 \) is a unit. We inductively define a sequence \( (g_i)_{i \in \mathbb{N}_0} \) by setting \( g_0 = f_0^{-1} \), and for \( i > 0 \),

\[
g_i = -f_0^{-1} \left( \sum_{j=1}^{i} f_j g_{i-j} \right).
\]

Evidently, each \( g_i \) is either zero or a homogeneous polynomial of degree \( i \) in \( R[X_1, \ldots, X_n] \). Also, we have \( f_0g_0 = 1 \) and for \( i > 0 \),

\[
0 = \sum_{j=0}^{i} f_j g_{i-j}.
\]

Now define

\[
g = \sum_{i=0}^{\infty} g_i.
\]

Then \( g \) is in \( R[[X_1, \ldots, X_n]] \), and this is the standard representation of \( g \). Using the above formula for \( fg \) we see that \( fg = 1 \).

### Assignment 3

**2.22** Let \( K \) be a field. Show that the ideal \( (X_1, X_2) \) of the commutative ring \( K[X_1, X_2] \) (of polynomials over \( K \) in indeterminates \( X_1, X_2 \)) is not principal.

**Suggest solution:** Assume that \( (X_1, X_2) = (f) \) for some \( f \in K[X_1, X_2] \); we will obtain a contradiction. Since \( X_1, X_2 \in (f) \), there exist \( g_1, g_2 \in K[X_1, X_2] \) such that

\[
X_1 = g_1f, \quad X_2 = g_2f.
\]

Applying the degree function to the first equation we obtain

\[
\deg(X_1) = \deg(g_1f)
\]

\[
1 = \deg(g_1) + \deg(f).
\]

Similarly,

\[
1 = \deg(g_2) + \deg(f).
\]

Since \( \deg(f), \deg(g_1), \) and \( \deg(f) \) are in \( \mathbb{N}_0 \), we must have \( \deg(f) = 0 \) or \( \deg(f) = 1 \). Assume first that \( \deg(f) = 0 \). Then \( f \in K \). Moreover, since \( f \neq 0 \) (otherwise \( X_1 = 0 \) and \( X_2 = 0 \), which is impossible), \( f \) is a unit in \( K \) and hence a unit in \( K[X_1, X_2] \). Now \( f \in (X_1, X_2) \). Hence, there exist
$h_1, h_2 \in K[X_1, X_2]$ such that

$$f = h_1X_1 + h_2X_2.$$ Evaluating both sides at $X_1 = 0$ and $X_2 = 0$, we obtain $f = 0$, a contradiction (recall that we just showed that $f$ is a non-zero constant). Hence, $\deg(f) = 1$. It follows that $\deg(g_1) = \deg(g_2) = 0$, so that $g_1, g_2 \in K$. Again, we see that $g_1$ and $g_2$ are non-zero and are hence units in $K$ and hence units in $K[X_1, X_2]$. Now

$$X_1 = g_1f = g_1g_2^{-1}g_2f = g_1g_2^{-1}X_2.$$ That is,

$$X_1 = (g_1g_2^{-1})X_2.$$ Evaluating both sides at $X_1 = 1$ and $X_2 = 0$, we obtain $1 = 0$, a contradiction.

2.30 Let $I, J$ be ideals of the commutative ring $R$. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$ Let $r \in \sqrt{IJ}$. Then there exists $n \in \mathbb{N}$ such that $r^n \in IJ$. Since $IJ \subseteq I \cap J$ we have $r^n \in I \cap J$. Hence, $r \in \sqrt{I \cap J}$. It follows that

$$\sqrt{IJ} \subseteq \sqrt{I \cap J}.$$ Let $r \in \sqrt{I \cap J}$. Then there exists $n \in \mathbb{N}$ such that $r^n \in I \cap J$. Since $I \cap J \subseteq I$ and $I \cap J \subseteq J$ we have $r \in \sqrt{I}$ and $r \in \sqrt{J}$. Thus, $r \in \sqrt{I} \cap \sqrt{J}$. It follows that

$$\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{I \cap J}.$$ Let $r \in \sqrt{I} \cap \sqrt{J}$. Then there exist $m, n \in \mathbb{N}$ such that $r^m \in I$ and $r^n \in J$. Hence, $r^{mn} = r^m r^n \in IJ$ so that $r \in \sqrt{IJ}$. It follows that

$$\sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}.$$ We have proven that

$$\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ}.$$ This implies that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$ Assignment 4

3.29 Determine the prime ideals of the ring $\mathbb{Z}/60\mathbb{Z}$ of residue classes of integers modulo 60.

Suggest solution: By 3.28, every prime ideal of $\mathbb{Z}/60\mathbb{Z}$ is of the form $P/60\mathbb{Z}$ where $P$ is a prime ideal of $\mathbb{Z}$ such that $60\mathbb{Z} \subseteq P$. By 3.34, every prime ideal $P$ of $\mathbb{Z}$ such that $60\mathbb{Z} \subseteq P$ is of the form $P = p\mathbb{Z}$, where $p$ is a prime of $\mathbb{Z}$ such that $60\mathbb{Z} \subseteq p\mathbb{Z}$, i.e., $p \mid 60$. It follows that the prime ideals of $\mathbb{Z}/60\mathbb{Z}$ are $2\mathbb{Z}/60\mathbb{Z}$, $3\mathbb{Z}/60\mathbb{Z}$, and $5\mathbb{Z}/60\mathbb{Z}$. 

7
3.31 Let $R$ be an integral domain. Recall that for $a_1, \ldots, a_n \in R$, where $n \in \mathbb{N}$, a greatest common divisor (GCD for short) or highest common factor of $a_1, \ldots, a_n$ is an element $d$ of $R$ such that

(i) $d \mid a_i$ for all $i = 1, \ldots, n$, and

(ii) whenever $c \in R$ is such that $c \mid a_i$ for all $i = 1, \ldots, n$, then $c \mid d$.

Show that every non-empty finite set of elements in a PID has a GCD.

Suggest solution: Assume that $R$ is a PID, and let $a_1, \ldots, a_n \in R$. Consider the ideal $(a_1, \ldots, a_n)$. Since $R$ is a PID, there exists $d \in R$ such that $(a_1, \ldots, a_n) = (d)$. We claim that $d$ is a GCD of $a_1, \ldots, a_n$. Since $a_1, \ldots, a_n \in (a_1, \ldots, a_n) = (d)$, we see that $d \mid a_i$ for $i = 1, \ldots, n$. Assume that $c \in R$ is such that $c \mid a_i$ for $i = 1, \ldots, n$. Let $r_i \in R$ be such that $a_i = r_i c$ for $i = 1, \ldots, n$. Also, let $x_1, \ldots, x_n$ be such that $x_1 a_1 + \cdots + x_n a_n = d$; note that $x_1, \ldots, x_n$ exist because $d \in (a_1, \ldots, a_n)$. Then

$$d = x_1 a_1 + \cdots + x_n a_n = x_1 r_1 c + \cdots + x_n r_n c = (x_1 r_1 + \cdots + x_n r_n) c.$$ 

Thus, $c \mid d$.

3.42 Show that an irreducible element in a unique factorization domain $R$ generates a prime ideal of $R$.

Suggest solution: Let $r \in R$ be irreducible. Then by definition $r$ is non-zero and not a unit. Since $r$ is not a unit we have $(r) \not\subseteq R$ (otherwise, $1 \in (r)$ so that $r$ is a unit). Let $a, b \in R$ be such that $ab \in (r)$; to prove that $(r)$ is a prime ideal it will suffice to prove that $a \in (r)$ or $b \in (r)$. If $a = 0$ or $b = 0$, then clearly $a \in (r)$ or $b \in (r)$; we may thus assume that $a \neq 0$ and $b \neq 0$. If $a$ or $b$ is a unit, then also $a \in (r)$ or $b \in (r)$; we may thus also assume that $a$ and $b$ are non-units. Since $ab \in (r)$, there exists $s \in R$ such that $ab = rs$. Since $R$ is an integral domain we have $rs = ab \neq 0$; also, $rs$ is not a unit (otherwise $(r)$ contains a unit, contradicting $(r) \not\subseteq R$). As $R$ is a UFD, there exist irreducible elements $p_1, \ldots, p_k, q_1, \ldots, q_\ell, y_1, \ldots, y_n$ in $R$ such that

$$a = p_1 \cdots p_k, \quad b = q_1 \cdots q_\ell, \quad rs = y_1 \cdots y_n.$$ 

Since $r$ is irreducible, we may assume that $y_1 = vr$ for some unit $v$ in $R$. Since $ab = rs$ we have

$$p_1 \cdots p_k q_1 \cdots q_\ell = vry_2 \cdots y_n.$$ 

Since $r$ is irreducible and $R$ is a UFD, there exists a unit $u$ in $R$ such that $r = up_i$ for some $i \in \{1, \ldots, k\}$ or $r = uq_j$ for some $j \in \{1, \ldots, \ell\}$. Hence, $p_i \in (r)$ for some $i \in \{1, \ldots, k\}$ or $q_j \in (r)$ for some $j \in \{1, \ldots, \ell\}$ (recall that $u$ is a unit, so that $p_i = u^{-1} r$ or $q_j = u^{-1} r$). This implies that $a \in (r)$ or $b \in (r)$, as desired.

3.47 Let $P$ be a prime ideal of the commutative ring $R$. Show that $\sqrt{P^n} = P$ for all $n \in \mathbb{N}$.

Suggest solution: Let $n \in \mathbb{N}$. Let $x \in \sqrt{P^n}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in P^n$. Now $P^n \subseteq P$. Hence, $x^m \in P$. Since $P$ is prime we have $x \in P$. This proves that $\sqrt{P^n} \subseteq P$. Let $x \in P$. Then $x^n \in P^n$. Therefore, $x \in \sqrt{P^n}$. This proves that $P \subseteq \sqrt{P^n}$. We conclude that $P = \sqrt{P^n}$. 

8
Assignment 5

3.50 Let $R$ be a commutative ring, and let $N$ be the nilradical of $R$. Show that the ring $R/N$ has zero nilradical.

**Suggest solution:** Let $x \in R/N$ and assume that $n \in \mathbb{N}$ is such that $x^n = 0_{R/N}$; we need to prove that $x = 0_{R/N}$. Let $a \in R$ be such that $x = a + N$. Then $0_{R/N} = x^n = (a + N)^n = a^n + N$. This means that $N = a^n + N$ so that $a^n \in N$. Since $a^n \in N$ there exists $m \in \mathbb{N}$ such that $(a^n)^m = 0$, i.e., $a^{nm} = 0$. Therefore, $a \in N$. We now have $x = a + N = N = 0_{R/N}$, as desired.

3.51 Let $R$ be a non-trivial commutative ring. Show that $R$ has exactly one prime ideal if and only if each element of $R$ is either a unit or nilpotent.

**Suggest solution:** Assume that $R$ has exactly one prime ideal $P$. Let $x \in R$. Assume $x$ is not a unit; we need to prove that $x$ is nilpotent. Since $x$ is not a unit $(x)$ is a proper ideal, and is hence included in a maximal ideal; since every maximal ideal is prime and $P$ is unique, $(x) \subseteq P$. Now by 3.49 we have

$$\sqrt{0} = \bigcap_{P' \in \text{Spec}(R)} P' = \bigcap_{P' \subseteq \{P\}} P' = P.$$ 

Hence, $x \in (x) \subseteq P = \sqrt{0}$. This implies that $x$ is nilpotent.

Next, assume that every element of $R$ is either a unit or nilpotent. Since $R$ is non-trivial, $0 \neq 1$. Hence, the ideal $0 = (0)$ is a proper ideal. Since 0 is proper, the ideal 0 is included in a maximal ideal; since every maximal ideal is prime, this proves that $R$ has at least one prime ideal. Let $P$ be a prime ideal of $R$; we will prove that $P = \sqrt{0}$, which will show that $P$ is unique. Let $r \in P$. Since $P$ is proper the element $r$ is not a unit. Hence, $r$ is nilpotent so that $r \in \sqrt{0}$. This proves that $P \subseteq \sqrt{0}$. Conversely, let $r \in \sqrt{0}$. Let $n \in \mathbb{N}$ be such that $r^n = 0$. Then $r^n = 0 \in P$. Since $P$ is prime we have $r \in P$. It follows that $\sqrt{0} \subseteq P$. We conclude that $P = \sqrt{0}$ so that $P$ is unique.

3.53 Let $P, I$ be ideals of the commutative ring $R$ with $P$ prime and $I \subseteq P$. Show that the non-empty set

$$\Theta = \{P' \in \text{Spec}(R) : I \subseteq P' \subseteq P\}$$

has a minimal member with respect to inclusion.

**Suggest solution:** We partially order $\Theta$ by declaring that $P_1 \leq P_2$ if and only if $P_2 \subseteq P_1$. The set $\Theta$ is non-empty because $P \in \Theta$. Let $Y$ be a totally ordered non-empty subset of $\Theta$; we need to prove that $Y$ has an upper bound in $\Theta$. Let $Q$ be the intersection of all the elements of $Y$. We claim that $Q \in \Theta$. Evidently, $Q$ is an ideal because $Q$ is the intersection of ideals. Also, it is clear that $I \subseteq Q \subseteq P$; in particular, $Q$ is proper because $P$ is proper. Let $a, b \in R$ be such that $ab \in Q$. Assume that $a \notin Q$; to prove that $Q$ is prime it will suffice to prove that $b \in Q$. Let $P' \in Y$; to prove that $b \in Q$ it will suffice to prove that $b \in P'$. Now since $a \notin Q$ there exists $P'' \in Y$ such that $a \notin P''$. Consider $P'$ and $P''$. Since $Y$ is totally ordered we have $P' \subseteq P''$ or $P'' \subseteq P'$. Assume first that $P' \subseteq P''$. Now $ab \in Q \subseteq P'$. Since $P'$ is prime we have $a \in P'$ or $b \in P'$. We cannot have $a \in P'$ for otherwise $a \in P' \subseteq P''$, contradicting $a \notin P''$. Therefore, $b \in P'$. Assume now that $P'' \subseteq P'$. We have $ab \in Q \subseteq P''$. Since $P''$ is prime we have $a \in P''$ or $b \in P''$. However, $a' \notin P''$;
hence, \( b \in P'' \subseteq P' \). We have proven that \( b \in P' \); thus, \( Q \) is a prime ideal of \( R \). It follows now that \( Q \in \Theta \). Clearly, \( Q \) is an upper bound in \( \Theta \) for \( Y \). By Zorn’s Lemma the set \( \Theta \) has a minimal member with respect to inclusion.

4.7 Let \( f : R \to S \) be a surjective homomorphism of commutative rings. Use the extension and contraction notation of 2.41 and 2.45 in conjunction with \( f \). Note that, by 2.46, \( C_R = \{ I \in \mathcal{I}_R : \ker(f) \subseteq I \} \) and \( \mathcal{E}_S = \mathcal{I}_S \). Let \( I \in C_R \). Show that

(i) \( I \) is a primary ideal of \( R \) if and only if \( I^e \) is a primary ideal of \( S \).

(ii) When this is the case, \( \sqrt{I} = (\sqrt{I^e})^c \) and \( \sqrt{T^e} = (\sqrt{T})^e \).

Suggest solution: We first note that by 2.46 we have \( J^e = f(J) \) for \( J \in C_R \), and also the maps

\[
C_R \xrightarrow{\text{extension}} \mathcal{I}_S \quad \text{and} \quad C_R \xleftarrow{\text{contraction}} \mathcal{I}_S
\]

are inverses of each other.

(i) Define \( g : R \to S/I^e = S/f(I) \) by \( g(r) = f(r) + f(I) \). It is straightforward to verify that \( g \) is a ring homomorphism. Since \( f \) is surjective, \( g \) is also surjective. Also, for \( r \in R \) we have

\[
g(r) = 0 \iff f(r) + f(I) = f(I)
\]

\[
\iff \text{there exists } x \in I \text{ such that } f(r) = f(x)
\]

\[
\iff \text{there exists } x \in I \text{ such that } f(r - x) = 0
\]

\[
\iff \text{there exists } x \in I \text{ such that } r - x \in \ker(f)
\]

\[
\iff r \in I \quad (\text{because } \ker(f) \subseteq I).
\]

Thus, \( \ker(g) = I \). By the Isomorphism Theorem, \( g \) induces an isomorphism of rings

\[
R/I \xrightarrow{\sim} S/f(I).
\]

Since \( R/I \) and \( S/f(I) \) are isomorphic the ideal \( I \) is primary if and only if \( f(I) \) is primary (see 4.3).

(ii) We first prove that \( \sqrt{I^e} = (\sqrt{I})^e \). Since \( I^e = f(I) \) and \( (\sqrt{T})^e = f(\sqrt{T}) \), we need to prove that \( \sqrt{f(I)} = f(\sqrt{I}) \). Let \( s \in \sqrt{f(I)} \). Let \( r \in R \) be such that \( f(r) = s \). Since \( s \in \sqrt{f(I)} \), there exists \( n \in \mathbb{N} \) such that \( s^n \in f(I) \). Let \( a \in I \) be such that \( s^n = f(a) \). We now have \( f(r^n - a) = 0 \). Since \( \ker(f) \subseteq I \), this implies that \( r^n \in I \). That is, \( r \in \sqrt{I} \). Applying \( f \), we obtain \( s = f(r) \in f(\sqrt{I}) \).

We have proven that \( \sqrt{f(I)} \subseteq f(\sqrt{I}) \). Next, let \( s \in f(\sqrt{I}) \). Let \( r \in \sqrt{I} \) be such that \( f(r) = s \). Since \( r \in \sqrt{I} \) there exists \( n \in \mathbb{N} \) such that \( r^n \in I \). Therefore, \( s^n = f(r^n) \in f(I) \). This implies that \( s \in \sqrt{f(I)} \), so that \( f(\sqrt{I}) \subseteq \sqrt{f(I)} \). Hence, \( \sqrt{f(I)} = f(\sqrt{I}) \).

Now

\[
(\sqrt{I^e})^c = (\sqrt{f(I)})^c \quad (\text{because } I^e = f(I))
\]

\[
= (f(\sqrt{I}))^c \quad (\text{by } \sqrt{f(I)} = f(\sqrt{I}))
\]

\[
= \sqrt{T} \quad (\text{by } 2.46; \text{see the above summary}).
\]
Let $I$ be a proper ideal of the commutative ring $R$, and let $P$ and $Q$ be ideals of $R$ which contain $I$. Prove that $Q$ is a $P$-primary ideal of $R$ if and only if $Q/I$ is a $P/I$-primary ideal of $R/I$.

Suggest solution: It will suffice to prove that $Q$ is primary if and only if $Q/I$ is primary and that $\sqrt{Q/I} = \sqrt{Q}/I$. Let $f : R \to R/I$ be the natural map. Then $f$ is a surjective ring homomorphism. By 4.7 (i), we have $Q$ is primary if and only if $f(Q) = Q/I$ is primary. It remains to prove that $\sqrt{Q/I} = \sqrt{Q}/I$. Now

\[
\sqrt{Q/I} = \sqrt{f(Q)} = \sqrt{Q}/I.
\]

Assignment 6

4.21 Let $f : R \to S$ be a homomorphism of commutative rings, and use the contraction notation of 2.41 in conjunction with $f$. Let $I$ be a decomposable ideal of $S$.

(i) Let

\[ I = Q_1 \cap \cdots \cap Q_n \quad \text{with} \quad \sqrt{Q_i} = P_i \quad \text{for} \quad i = 1, \ldots, n \]

be a primary decomposition of $I$. Show that

\[ I^c = Q_1^c \cap \cdots \cap Q_n^c \quad \text{with} \quad \sqrt{Q_i^c} = P_i^c \quad \text{for} \quad i = 1, \ldots, n \]

is a primary decomposition of $I$. Deduced that $I^c$ is a decomposable ideal of $R$ and that

\[ \text{ass}_R(I^c) \subseteq \{ P^c : P \in \text{ass}_R(I) \}. \]

(ii) Now assume that $f$ is surjective. Show that, if the first primary decomposition in (i) is minimal, then so too is the second, and deduce that in these circumstances,

\[ \text{ass}_R(I^c) = \{ P^c : P \in \text{ass}_R(I) \}. \]

Suggest solution: (i) We have

\[
I^c = f^{-1}(I) = f^{-1}(Q_1 \cap \cdots \cap Q_n) = f^{-1}(Q_1) \cap \cdots \cap f^{-1}(Q_n) = Q_1^c \cap \cdots \cap Q_n^c.
\]

Also, for $i \in \{1, \ldots, n\}$,

\[
\sqrt{Q_i^c} = (\sqrt{Q_i})^c \quad \text{(2.43(iv))}
\]
Next we prove that $Q_i^c$ is primary for $i \in \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$. The ideal $Q_i^c$ is primary (otherwise, $1 \in Q_i^c$ so that $1 = f(1) \in Q_i$, a contradiction). Let $a, b \in R$ and assume that $ab \in Q_i^c$ and $a \notin Q_i^c$; we need to prove that $b \in \sqrt{Q_i^c}$. Since $ab \in Q_i^c = f^{-1}(Q_i)$ we have $f(ab) = f(a)f(b) \in Q_i$. Since $Q_i$ is primary, we have $f(a) \in Q_i$ or $f(b) \in \sqrt{Q_i}$. If $f(a) \in Q_i$, then $a \in f^{-1}(Q_i) = Q_i^c$, a contradiction. Hence, $f(b) \in \sqrt{Q_i}$. This means that $b \in f^{-1}(\sqrt{Q_i}) = (\sqrt{Q_i})^c = \sqrt{Q_i^c}$. Hence, $Q_i^c$ is primary. This completes the proof that the above is a primary decomposition of $I^c$ and thus $I^c$ is decomposable. We have $\text{ass}_R(I^c) \subseteq \{P^c : P \in \text{ass}_R(I)\}$ because the above primary decomposition can be refined to a minimal primary decomposition (see 4.16 or the lecture notes).

(ii) Assume that $f$ is surjective. Assume that the first primary decomposition in (i) is minimal; we need to prove that second primary decomposition is also minimal. First we verify that $P_1^c, \ldots, P_n^c$ are pairwise unequal. Assume that $P_i^c = P_j^c$ for some $i, j \in \{1, \ldots, n\}$. Then $f^{-1}(P_i) = f^{-1}(P_j)$. Applying $f$ and using that $f$ is surjective, we find that $P_i = P_j$. As the first primary decomposition is minimal, we must have $i = j$. This implies that $P_1^c, \ldots, P_n^c$ are pairwise unequal. Finally, assume that $i \in \{1, \ldots, n\}$ is such that

$$\bigcap_{j=1 \atop j \neq i}^n Q_j^c \subseteq Q_i^c.$$ 

Let $y \in \bigcap_{j=1 \atop j \neq i}^n Q_j$. Since $f$ is surjective, there exists $x \in R$ such that $f(x) = y$. Since $y \in Q_j$ for $j \neq i$, we have $x \in f^{-1}(Q_j) = Q_j^c$ for $j \neq i$. Therefore, $x \in \bigcap_{j=1 \atop j \neq i}^n Q_j^c$. By the assumed inclusion, we get $x \in Q_i^c = f^{-1}(Q_i)$. This implies that $y \in Q_i$. We have proven that

$$\bigcap_{j=1 \atop j \neq i}^n Q_j \subseteq Q_i,$$

contradicting the minimality of the first primary decomposition. That $\text{ass}_R(I^c) = \{P^c : P \in \text{ass}_R(I)\}$ follows from definition of $\text{ass}_R(I^c)$.

4.22 Let $f : R \to S$ be a surjective homomorphism of commutative rings; use the extension notation of 2.41 in conjunction with $f$. Let $I, Q_1, \ldots, Q_n, P_1, \ldots, P_n$ be ideals of $R$ that contain $\ker(f)$. Show that

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with} \quad \sqrt{Q_i} = P_i \quad \text{for} \quad i = 1, \ldots, n$$

(1)

is a primary decomposition of $I$ if and only if

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with} \quad \sqrt{Q_i^e} = P_i^e \quad \text{for} \quad i = 1, \ldots, n$$

(2)

is a primary decomposition of $I^e$, and that, when this is the case, the first of these is minimal if and only if the second is. Deduce that $I$ is a decomposable ideal of $R$ if and only if $I^e$ is a decomposable
ideal of $S$, and when this is the case,

$$\text{ass}_R(I) = \{P^e : P \in \text{ass}_R(I)\}.$$ 

**Suggest solution:** We first note the following fact: if $A$ and $B$ are ideals of $R$ such that $\ker(f) \subseteq A$ and $\ker(f) \subseteq B$, then $f(A \cap B) = f(A) \cap f(B)$. We leave the proof of this as an exercise. Assume that (1) is a primary decomposition. Then

$$I = Q_1 \cap \cdots \cap Q_n$$
$$f(I) = f(Q_1 \cap \cdots \cap Q_n)$$
$$I^e = f(Q_1) \cap \cdots \cap f(Q_n)$$
$$I^e = Q_1^e \cap \cdots \cap Q_n^e.$$ 

Also, if $i \in \{1, \ldots, n\}$, then $f(Q_i) = Q_i^e$ is primary and $\sqrt{Q_i^e} = P_i^e$ by 4.7. Thus, (2) is a primary decomposition. Assume that (1) is a minimal primary decomposition; we want to prove that (2) is also a minimal primary decomposition. We first prove that $P_1^e, \ldots, P_n^e$ are pairwise unequal. Assume that $P_i^e = P_j^e$ for some $i, j \in \{1, \ldots, n\}$; we need to prove $i = j$. Now since $P_i^e = P_j^e$ we have $(P_i^e)^c = (P_j^e)^c$. Now $(P_i^e)^c = f^{-1}(f(P_i)) = P_i$ because $f$ is surjective and $\ker(f) \subseteq P_i$; similarly, $(P_j^e)^c = P_j$. We thus get $P_i = P_j$. Since (1) is minimal we must have $i = j$. 

Finally, assume that $i \in \{1, \ldots, n\}$ is such that

$$\bigcap_{j=1}^n Q_j^e \subseteq Q_i^e;$$

we will obtain a contradiction. Now

$$\bigcap_{j=1}^n Q_j^e \subseteq Q_i^e$$

$$f^{-1}(\bigcap_{j=1}^n Q_j^e) \subseteq f^{-1}(Q_i^e)$$

$$\bigcap_{j=1}^n f^{-1}(Q_j^e) \subseteq Q_i$$

$$\bigcap_{j=1}^n Q_j \subseteq Q_i.$$ 

This contradicts that (1) is a minimal primary decomposition.
Next, 4.21 implies that if \( (2) \) is a primary decomposition, then \( (1) \) is a primary decomposition, and also if \( (2) \) is a minimal primary decomposition, then \( (1) \) is a minimal primary decomposition. The remaining assertion follows immediately from what we have already proven.

4.28 Let \( K \) be a field and let \( R = K[X,Y] \) be the ring of polynomials over \( K \) in indeterminates \( X, Y \). In \( R \), let \( I = (X^3, XY) \).

(i) Show that, for every \( n \in \mathbb{N} \), the ideal \( (X^3, XY, Y^n) \) of \( R \) is primary.

(ii) Show that \( I = (X) \cap (X^3, Y) \) is a minimal primary decomposition of \( I \).

(iii) Construct infinitely many different minimal primary decompositions of \( I \).

Suggest solution: (i) Let \( M = (X, Y) \). For \( n \in \mathbb{N} \) let \( I_n = (X^3, XY, Y^n) \). We have

\[
M^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq I_1 = (X^3, XY, Y) \subseteq M = (X, Y),
\]
\[
M^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq I_2 = (X^3, XY, Y^2) \subseteq M = (X, Y)
\]

and if \( n \geq 3 \),

\[
M^n = (X^n, X^{n-1}Y, \ldots, XY^{n-1}, Y^n) \subseteq I_n = (X^3, XY, Y^n) \subseteq M = (X, Y).
\]

Taking radicals, we obtain

\[
\sqrt{M^3} = M \subseteq \sqrt{I_1} \subseteq \sqrt{M} = M,
\]
\[
\sqrt{M^3} = M \subseteq \sqrt{I_2} \subseteq \sqrt{M},
\]

and if \( n \geq 3 \),

\[
\sqrt{M^n} = M \subseteq \sqrt{I_n} \subseteq \sqrt{M} = M.
\]

It follows that \( \sqrt{I_n} = M \) for all \( n \in \mathbb{N} \). By Proposition 4.9 the ideal \( I_n \) is primary for all \( n \in \mathbb{N} \).

(ii) First we prove that \( I = (X) \cap (X^3, Y) \). It is clear that \( I \subseteq (X) \cap (X^3, Y) \). Let \( g \in (X) \cap (X^3, Y) \). Then there exist \( a, b, c \in R \) such that \( g = aX \) and \( g = bX^3 + cY \). Now \( aX = bX^3 + cY \). Substituting \( X = 0 \) we obtain \( 0 = c(0, Y)Y^3 \). This implies that there exists \( d \in R \) such that \( c = dX \). We now have \( g = bX^3 + dXY \). Hence, \( g \in I \) so that \( (X) \cap (X^3, Y) \subseteq I \). It follows that \( I = (X) \cap (X^3, Y) \).

Next, we note that \( (X) \) is a prime ideal of \( R \) (since \( R/(X) \cong K[Y] \), which is an integral domain). Also, we have

\[
(X,Y)^3 = (X^3, X^2Y, XY^2, Y^3) \subseteq (X, Y) \subseteq (X,Y).
\]

Taking radicals, we obtain

\[
(X,Y) \subseteq \sqrt{(X^3, Y)} \subseteq (X,Y).
\]

Hence, \( (X,Y) = \sqrt{(X^3, Y)} \), which implies by Proposition 4.9 that \( (X^3, Y) \) is primary (since \( (X,Y) \) is maximal). It is clear that the primary decomposition \( I = (X) \cap (X^3, Y) \) is minimal.

(iii) Using the method of (ii) we find that

\[
I = (X^3, XY) = (X) \cap (X^3, XY, Y^n)
\]
for $n \in \mathbb{N}$. The ideal $(X)$ is prime and primary, and $(X^3, XY, Y^n)$ is primary with radical $(X, Y)$ for $n \in \mathbb{N}$ by (i). Hence, this is a primary decomposition of $I$. It is straightforward to verify that this primary decomposition is minimal. The primary decompositions $I = (X) \cap (X^3, XY, Y^n)$ are all different because $(X^3, XY, Y^n) \neq (X^3, XY, Y^m)$ for $m, n \in \mathbb{N}$ with $m \neq n$.

**Assignment 8**

5.26. Let the situation be as in 5.23. Show that if the ring $R$ is Noetherian, then so too is the ring $S^{-1}R$.

**Suggest solution:** Assume that $R$ is Noetherian. Let

$$J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$$

be a sequence of ideals in $S^{-1}R$. Then

$$(J_1)^c \subseteq (J_2)^c \subseteq (J_3)^c \subseteq \cdots$$

is a sequence of ideals in $R$. Since $R$ is Noetherian, there exists $n \in \mathbb{N}$ such that for $k \in \mathbb{N}$ with $k \geq n$ we have $(J_{n+k})^c = (J_n)^c$. Therefore, $((J_{n+k})^c)^c = ((J_n)^c)^c$ for $k \geq n$. By 5.24 we have $((J_n)^c)^c$ and $((J_{n+k})^c)^c = J_{n+k}$ for $k \geq n$. Hence, $J_{n+k} = J_n$ for $k \geq n$. It follows that $S^{-1}R$ is Noetherian.

**Suggest solution:** Alternatively, we can argue as follows. Assume that $R$ is Noetherian. Assume that $J$ is an ideal of $S^{-1}R$; to prove that $S^{-1}R$ is Noetherian, it will suffice to prove that $J$ is finitely generated. Then $J^c$ is an ideal of $R$. Since $R$ is Noetherian, $J^c$ is finitely generated by, say, $r_1, \ldots, r_t$: $J^c = (r_1, \ldots, r_t)$. We claim that $(J^c)^c$ is generated by $r_1/1, \ldots, r_t/1$. It is clear that $r_1/1, \ldots, r_t/1$ are contained in $(J^c)^c$. Let $x \in (J^c)^c$. By 5.25 there exist $a \in J^c$ and $s \in S$ such that $x = a/s$. Since $a \in J^c$ there exist $c_1, \ldots, c_t \in R$ such that $a = c_1 r_1 + \cdots + c_t r_t$. This implies that

$$x = a/s$$

$$= (c_1 r_1 + \cdots + c_t r_t)/s$$

$$= c_1 r_1/s + \cdots + c_t r_t/s$$

$$= (c_1/s)(r_1/1) + \cdots + (c_t/s)(r_t/1).$$

Thus, $x \in (r_1/1, \ldots, r_t/1)$. We have proven that $(J^c)^c = (r_1/1, \ldots, r_t/1)$, so that $(J^c)^c$ is finitely generated. Since $J = (J^c)^c$ by 5.24, $J$ is finitely generated. This implies that $S^{-1}R$ is Noetherian.

5.34. Let $R$ be a non-trivial commutative ring, and assume that, for each $P \in \text{Spec}(R)$, the localization $R_P$ has no non-zero nilpotent element. Show that $R$ has no non-zero nilpotent element.

**Suggest solution:** Assume that $x \in R$ is such that $x \neq 0$ and $x$ is nilpotent; we will obtain a contradiction. Let $I = \{s \in R : sz = 0\}$. Then $I = (0 : x)$, and $I$ is an ideal of $R$. Assume that $I = R$. Then $1 \in I$; this implies that $1 \cdot x = 0$, i.e., $x = 0$; this is a contradiction. Hence, $I \subsetneq R$. 

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Since $I$ is a proper ideal, $I$ is included in a maximal ideal $M$. Since $M$ is a maximal ideal, $M$ is prime. Consider $R_M$ and the image $x/1$ of $x$ in $R_M$ under the natural map $R \rightarrow R_M$. Since $x$ is nilpotent so is $x/1$. By hypothesis, $R_M$ does not contain a non-zero nilpotent element. Therefore, $x/1 = 0_{R_M} = 0/1$. This implies that there exists an element $s \in S = R - M$ such that $sx = 0$. By the definition of $I$ we have $s \in I \subseteq M$. We now have $s \in M \cap (R - M)$; this is a contradiction.

**6.11.** Let $M$ be a module over a commutative ring $R$, and let $J \subseteq M$; let $G$ be the submodule of $M$ generated by $J$.

(i) Show that, if $J = \emptyset$, then $G = 0$.

(ii) Show that, if $J \neq \emptyset$, then

$$G = \left\{ \sum_{i=1}^{n} r_i j_i : n \in \mathbb{N}, r_1, \ldots, r_n \in R, j_1, \ldots, j_n \in J \right\}.$$

(iii) Show that, if $\emptyset \neq J = \{l_1, \ldots, l_t\}$, then

$$G = \left\{ \sum_{i=1}^{t} r_i l_i : r_1, \ldots, r_t \in R \right\}.$$

**Suggest solution:**

(i) Assume that $J = \emptyset$. Since $G$ is a submodule of $M$ we have $0 \subseteq G$. Also, $0$ is a submodule of $M$ such that $\emptyset \subseteq 0$. This implies that

$$G = \bigcap_{N \text{ submodule of } M \text{ such that } J \subseteq N} N \subseteq 0.$$ 

Hence, $G = 0$.

(ii) Define

$$W = \left\{ \sum_{i=1}^{n} r_i j_i : n \in \mathbb{N}, r_1, \ldots, r_n \in R, j_1, \ldots, j_n \in J \right\}.$$

We need to prove that $G = W$. Using the submodule criterion, it is straightforward to verify that $W$ is a submodule of $M$ that contains $J$. Hence,

$$G = \bigcap_{N \text{ submodule of } M \text{ such that } J \subseteq N} N \subseteq W.$$ 

Since $G$ contains $J$, $G$ also contains all $R$-linear combinations of elements of $J$. Thus, $W \subseteq G$. We conclude that $G = W$.

(iii) Let $W$ be as above, and let

$$U = \left\{ \sum_{i=1}^{t} r_i l_i : r_1, \ldots, r_t \in R \right\}.$$
Evidently, $U \subseteq W$. Conversely, let $x = \sum_{i=1}^{n} r_{ij}i \in W$. Recalling that $J = \{l_1, \ldots, l_t\}$, we have:

$$
x = \sum_{i=1}^{n} r_{ij}i
= \left( \sum_{i=1}^{n} r_{ij}i \right) + \cdots + \left( \sum_{i=1}^{n} r_{ij}i \right)
= \left( \sum_{i=1}^{n} r_{lj}l_1 \right) + \cdots + \left( \sum_{i=1}^{n} r_{lj}l_t \right)
= \left( \sum_{i=1}^{n} r_i \right) l_1 + \cdots + \left( \sum_{i=1}^{n} r_i \right) l_t
\in U.
$$

Thus, $W \subseteq U$. It follows that $W = U$.

**Assignment 11**

**7.45** Let $G$ be a module over a non-trivial commutative Noetherian ring $R$. Show that $G$ has finite length if and only if $G$ is finitely generated and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \ldots, M_n$ of $R$ (not necessarily distinct) such that

$$
M_1 \cdots M_n G = 0.
$$

**Suggest solution:** Assume that $G$ has finite length. By 7.36 the $R$-module $G$ is Noetherian. By 7.13, $G$ is finitely generated. Let

$$
0 = G_0 \supsetneq G_1 \supsetneq \cdots \supsetneq G_{n-1} \supsetneq G_n = G
$$

be a composition series. By definition, $G_i/G_{i-1}$ is simple for $i = 1, \ldots, n$. By 7.32, for each $i \in \{1, \ldots, n\}$ there exists a maximal ideal $M_i$ of $R$ such that $G_i/G_{i-1} \cong R/M_i$ as $R$-modules. Now let $g \in G$, and let $m_i \in M_i$ for $i \in \{1, \ldots, n\}$. Since $G_n/G_{n-1} \cong R/M_n$, we have $r(x + G_{n-1}) = 0$ for $r \in M_n$ and $x \in G_n$. This implies that $m_n g \in G_{n-1}$. Similarly, $m_{n-1}m_ng \in G_{n-2}$, and continuing, we find that $m_1 \cdots m_ng \in G_0 = 0$. This proves that $M_1 \cdots M_n G = 0$.

Now assume that $G$ is finitely generated and and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \ldots, M_n$ of $R$ (not necessarily distinct) such that

$$
M_1 \cdots M_n G = 0.
$$
Since $R$ is a Noetherian ring (by assumption), and since $G$ is finitely generated, $G$ is Noetherian by 7.22. By 7.30, $G$ is also Artinian (this uses the hypothesis $M_1 \cdots M_n G = 0$). By 7.36, $G$ has finite length.

**7.46** Let $R$ be a principal ideal domain which is not a field. Let $G$ be an $R$-module. Show that $G$ has finite length if and only if $G$ is finitely generated and there exists $r \in R$ with $r \neq 0$ such that $rG = 0$.

Assume that $G$ has finite length. By 7.45, $G$ is finitely generated, and there exist $n \in \mathbb{N}$ and maximal ideals $M_1, \ldots, M_n$ of $R$ (not necessarily distinct) such that

$$M_1 \cdots M_n G = 0.$$ 

Since $R$ is not a field, 0 is not a maximal ideal of $R$. This implies that $M_1, \ldots, M_n$ are all non-zero. Since $R$ is a PID, we may write $M_i = (r_i)$ for some $r_i \in R$ for $i \in \{1, \ldots, n\}$. Since $M_1 \cdots M_n G = 0$ we have $rG = 0$ with $r = r_1 \cdots r_n$; note that $r \neq 0$ as $r_1 \neq 0, \ldots, r_n \neq 0$, and $R$ is an integral domain.

Now suppose that $G$ is finitely generated and there exists $r \in R$ with $r \neq 0$ and $rG = 0$. If $r$ is a unit, then $G = 0$, and $G$ has finite length. Assume that $r$ is not a unit. Since $R$ is a PID, $R$ is a UFD by 3.39. Therefore, there exist $n \in \mathbb{N}$ and irreducible elements $p_1, \ldots, p_n \in R$ such that $r = p_1 \cdots p_n$. Let $M_i = (p_i)$ for $i \in \{1, \ldots, n\}$. By 3.34, $M_i$ is a maximal ideal of $R$ for $i \in \{1, \ldots, n\}$. Since $rG = 0$ we have $M_1 \cdots M_n G = 0$. By 7.45 we now conclude that $G$ has finite length.