Minimum Sets Forcing Monochromatic Triangles

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Abstract. For a given triangle T, consider the problem of finding a finite set S in the plane such that every two-coloring of S results in a monochromatic set congruent to the vertices of T. We show that such a set S must have at least seven points. Furthermore we show by an example that the minimum of seven is achieved.

1. Introduction

The fundamental problem in Euclidean Ramsey theory is the following: Given a configuration C of points in $\mathbb{R}^n$ and an arbitrary k-coloring of $\mathbb{R}^n$, does there exist a monochromatic set of points in $\mathbb{R}^n$ congruent to C? In this paper we focus on the case where $k = n = 2$ and C is the vertex set of a triangle. We will say that a triangle T is *2-Ramsey* if every 2-coloring of $\mathbb{R}^2$ gives a monochromatic set congruent to the vertex set of T.

The foundations of Euclidean Ramsey theory were laid in a sequence of three seminal papers [2], [3], and [4]. Among the many results of these papers, the authors make the following conjecture:

Conjecture: All non-equilateral triangles are 2-Ramsey.

A simple 2-coloring scheme by alternating strips shows that equilateral triangles are *not* 2-Ramsey, and it is known that several classes of triangles, including all right triangles, *are* 2-Ramsey (see [4] and [8]). The conjecture above, however, remains open to the present. See [5] and section 6.3 of [1] for current surveys of the subject.

By an application of the compactness principle, if a triangle T is 2-Ramsey, then there must exist a finite set $S \subset \mathbb{R}^2$ such that every 2-coloring of S results in a monochromatic set congruent to the vertex set of T. For convenience in terminology we introduce the following notion.

Definition. Let T be a triangle, let $S \subset \mathbb{R}^2$ be a finite set, and let $\mathcal{T}$ be a collection of three-point subsets of S. Suppose further that each member of $\mathcal{T}$ forms the vertex set of a triangle congruent to T. Then we say that the pair $(S, \mathcal{T})$ is *T-forcing* if every 2-coloring of S results in a monochromatic member of $\mathcal{T}$.

Note that a T-forcing pair is combinatorially a 3-uniform hypergraph with chromatic number greater than two.

If follows (from our observation above concerning the compactness principle) that T is a 2-Ramsey triangle if and only if there is a T-forcing pair $(S, \mathcal{T})$. For this paper we investigate "minimizing" the pair $(S, \mathcal{T})$. In particular, consider the following natural questions.

Question 1. If T is a 2-Ramsey triangle, what is the value of
$p(T) = \min\{k : \text{there exists a } T\text{-forcing } (S, \mathcal{T}) \text{ with } |S| = k\}$?

Question 2. If T is a 2-Ramsey triangle, what is the value of
$m(T) = \min\{k : \text{there exists a } T\text{-forcing } (S, \mathcal{T}) \text{ with } |\mathcal{T}| = k\}$?

Question 3. What is the minimum of $p(T)$ over all 2-Ramsey triangles?

Question 4. What is the minimum of $m(T)$ over all 2-Ramsey triangles?

In this paper we answer question 3 by showing that the minimum is achieved by the triangle T^* with angles $\pi/7$, $2\pi/7$, and $4\pi/7$, with $p(T^*) = 7$. It is well-known (see p. 104 of [6]) that no 3-uniform hypergraph with fewer than seven hyperedges has chromatic number greater than two; in fact, our construction for T^*, taken as a hypergraph, yields the known minimal example. Thus, the answer to question 4 is seven as well, minimized by $m(T^*)$. It is interesting to note in this context that seven appears as an answer to at least one other related question in Euclidean Ramsey theory: there is a configuration of seven points in the plane such that every 3-coloring of that set results in two points of the same color at unit distance, but that no set of six points has this property [7].

Our main result is given in the next section. In section 3 we give the construction for T^* and conclude with some bounds for the case of the triangle with angles $\pi/6$, $\pi/3$, and $\pi/2$.

2. The main result

For convenience we adopt the following terminology: if T is a triangle and S is a set of points in the plane then we will refer to a three-point subset of S as a *copy of T in S* if it forms the vertex set of a triangle congruent to T.

Theorem. There is no triangle T with $p(T) \leq 6$.

Proof: Assume (to reach a contradiction) that T is a triangle, that $(S, \mathcal{T})$ is T-forcing, and that $S = \{A, B, C, D, E, F\}$. We may assume that $\mathcal{T}$ contains all copies of T in S. Let k be the maximum integer so that $\mathcal{T}$ contains k members sharing two points in common. Clearly we have $1 \leq k \leq 4$.

The case $k = 1$ is easily eliminated: if no segment determined by S belongs to more than one copy of T then S is easily two-colored so as to avoid monochromatic copies of T. Indeed, of the $\binom{6}{3}/2 = 10$ ways to partition S into two color sets of three points each, at most $\binom{6}{2}/3 = 5$ may feature a monochromatic copy of T.

The case $k = 4$ is also easily eliminated. For if AB belongs to four distinct copies of T then the set S must be arranged as in Figure 1, and the coloring shown in that figure avoids monochromatic copies of T. (It's easy to see that the monochromatic isosceles triangles in this coloring cannot be congruent to T.)

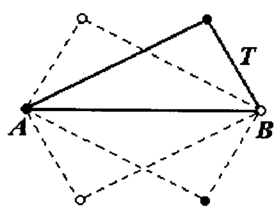

Figure 1.

For the case $k = 3$ assume that AB belongs to three copies of T in S, say ABC, ABD, and ABE. We may choose a coordinate system for $\mathbb{R}^2$ so that $A = (-1, 0)$, $B = (1, 0)$, $C = (u, v)$, $D = (u, -v)$, and $E = (-u, v)$ (with $u, v > 0$). Now by assumption we know that ABF is not congruent to T. Since S is a forcing set for T, CDE must be congruent to T. Thus we have

$$\begin{aligned} \{|CD|, |DE|, |CE|\} &= \{|AB|, |BC|, |AC|\} \\ \{2u, 2v, 2\sqrt{u^2 + v^2}\} &= \{2, \sqrt{(u-1)^2 + v^2}, \sqrt{(u+1)^2 + v^2}\} \end{aligned}$$

Neither of the possibilities $u = 1$ or $v = 1$ lead to solutions with positive values of u and v. The remaining possibility ($u^2 + v^2 = 1$) has only one feasible solution, namely $u = 1/2$ and $v = \sqrt{3}/2$. But this, of course, means that S consists of five vertices of a regular hexagon together with an additional sixth point. It is clear that the triangle T would have to be determined by three points of the hexagon, and it is easy to verify that this set is not a forcing set for any such triangle.

This leaves only the case $k = 2$. Now the six points of S determine 20 triangles, and since S is assumed to be a forcing set for T, if a three-point subset of S is not a copy of T, then its complement is. So, S must determine at least 10 copies of T. Thus the 15 segments determined by S must account for at least 30 sides of copies of T. Since $k = 2$ this means that each segment determined by S

must appear as a side of exactly two copies of T, and that S determines exactly 10 copies of T.

Now T must have only one longest side (for otherwise the six point set S would have to determine 10 segments of maximum length – an impossibility). Let the side lengths of T be $\alpha \leq \beta < \gamma$. Then S determines exactly five segments of length γ. This requires at least five of the points of S to be on the boundary of the convex hull of S. Observe that no segment of length γ may appear on the boundary of the convex hull of S. For if AB is on the boundary of the convex hull with $|AB| = \gamma$, and if ABC and ABD are copies of T, then the remaining two points of S must appear on the dashed arcs in Figure 2. But such placement does not allow for five segments of length γ.

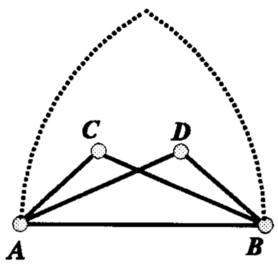

Figure 2.

We can now note that T must in fact be determined by three consecutive points on the boundary of the convex hull of S. For let A, B, and C be consecutive on this boundary and suppose $ABC \ncong T$. Then by coloring and $\{A, B, C\}$ black and $\{D, E, F\}$ white we see $DEF \cong T$, so we may assume that $|DF| = \gamma$. Then D and F lie on the convex hull's boundary, but cannot be consecutive by our previous observation. This must mean that D, E, and F are consecutive on the hull's boundary.

As we have noted, the convex hull of S must be a pentagon or a hexagon. First, consider the pentagonal case. If the convex hull of S is the pentagon $ABCDE$ then all five diagonals of this pentagon must have length γ. Since T has only one side of length γ, the two copies of T to which segment AB belongs must be ABC and ABE. This means that $|BC| = |AE|$, and repeating the argument leads to the conclusion that $ABCDE$ is a regular pentagon. But S determines only three distinct segment lengths, so the sixth point F must be the center of this pentagon. But it is easy to see that this configuration is not T-forcing for the triangle $T \cong ABC$.

It remains only to consider the case that all six points of S are on its convex hull. So suppose the convex hull of S is the hexagon $ABCDEF$. Now if $BDF \ncong T$ then (since S is a forcing set for T) we must have $ACE \cong T$. We may assume that $|CE| = \alpha$ so that $\angle CAE$ is a smallest angle in T. Then we must have both $CAD \ncong T$ and $EAD \ncong T$. From this it follows that $BCF \cong T$ and $BEF \cong T$ which accounts for both copies of T sharing side BF. This means $BFA \ncong T$ so that $DCE \cong T$. But since $|CE| = \alpha$ this means that either CD or DE has length γ, contradicting our previous note that no segment on the boundary of the convex hull may have length γ.

All cases have now been accounted for, so the proof is complete.

□

3. Examples

We conclude with two examples related to the questions in the introduction. The first example shows the minimum of $p(T) = 7$ in the above theorem is achieved.

Example 1. Let S be the set of vertices of a regular 7-gon labeled in order $0, 1, \ldots 6$ and let $\mathcal{T}^* = \{\{i, i+1, i+3\} : 0 \leq i \leq 6\}$, taking all integers modulo 7. (The reader may note that $(S, \mathcal{T}^*)$ constitutes a representation of the well-known Fano plane.) Let T^* be the triangle with vertex set $\{0, 1, 3\}$. We claim that $(S, \mathcal{T}^*)$ is T^*-forcing. To see this, first note that we may assume 0 and 1 are both colored black. There are now two cases, depending on the color given to point 2.

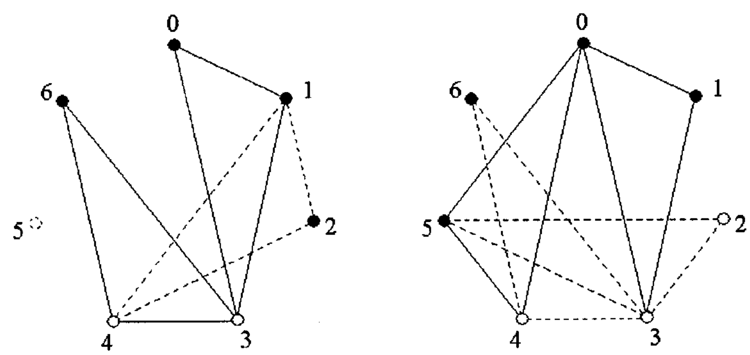

Figure 3.

- Suppose first that point 2 is also colored black and refer to the left half of Figure 3 above. Considering the sets $\{0, 1, 3\}$ and $\{1, 2, 4\}$ we see that we must color points 3 and 4 white to avoid a monochromatic member of $\mathcal{T}^*$. Likewise, then, taking $\{3, 4, 6\}$ into consideration forces point 6 to be colored black, leading to the monochromatic $\{6, 0, 2\} \in \mathcal{T}^*$.

- Now suppose point 2 is colored white and refer to the right half of Figure 3. The triple $\{0, 1, 3\}$ still forces 3 to be white, and then in turn $\{2, 3, 5\}$ forces

5 to be black. Then $\{4,5,0\}$ forces 4 to be white, and $\{3,4,6\}$ forces 6 to be black. This leads to the monochromatic set $\{5,6,1\} \in \mathcal{T}^*$.

□

The next example demonstrates bounds on $p(T')$ and $m(T')$ for a specific right triangle T'. We are indebted to our colleague C.O. Christenson for motivating this example.

Example 2. Let S be the nine point subset of the lattice of equilateral triangles shown in Figure 3, and let T' be the triangle AYB with angles $\pi/6$, $\pi/3$, and $\pi/2$. Clearly any two-coloring of S must result in one of $\{A,B\}$, $\{B,C\}$, or $\{A,C\}$ being monochromatic. But note that if $\{A,B,X,Y,Z\}$ (the points connected by bold segments in the figure) are two-colored so that A and B receive the same color, then one of the triangles ABX, ABY, ABZ, or XYZ will be monochromatic. Similarly, if B and C [A and C] receive the same color, then one of four triangles with vertices among the points joined by dashed [wavy] segments will be monochromatic. Thus, every two-coloring of S results in one of twelve copies of T' being monochromatic, so $p(T') \leq 9$ and $m(T') \leq 12$. □

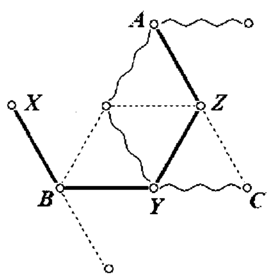

Figure 4.

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