On some numbers related to the Erdös-Szekeres Theorem

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Abstract. A *crossing family* of segments is a collection of segments each pair of which crosses. Given positive integers j and k, a (j, k) grid is the union of two pairwise-disjoint collections of segments (with j and k members, respectively) such that each segment in the first collection crosses all members of the other. Let $c(k)$ be the least integer such that any planar set of $c(k)$ points in general position generates a crossing family of k segments. Also let $\#(j, k)$ be the least integer such that any planar set of $\#(j, k)$ points in general position generates a (j, k)-grid. We establish here the facts $9 \leq c(3) \leq 16$ and $\#(1, 2) = 8$.

For each positive integer $n \geq 3$ let $g(n)$ be the least integer such that every planar set of $g(n)$ points in general position contains the vertices of some convex n-gon. This number was introduced by Erdös and Szekeres in 1935 (see [6] and [7]) who established the bounds $2^{n-2} + 1 \leq g(n) \leq \binom{2n-4}{n-2} + 1$ and conjectured that the lower bound is in fact an equality. The values $g(3) = 3$ and $g(4) = 5$ are easy, and several proofs of the fact $g(5) = 9$ have been given. However, no other values have been computed exactly and the upper bound given by Erdös and Szekeres stood until recently as the best known. A seqence of 1998 papers by Chung and Graham [2], Kleitman and Pachter [3], and finally Tóth and Valtr [8] improved the above-mentioned bound to

$$(0.1) \qquad 2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-2} + 2 \, .$$

Morris and Soltan [4] provide an excellent survey of related results.

Given the apparent difficulty of determining values of $g(n)$ we might well seek weakened notions of these numbers. For example, let $\mathcal{P}(n)$ be

a combinatorial property satisfied by the vertex set of a convex n-gon. It might be interesting to ask how large a set X must be to guarantee the existence of a subset of X having property $\mathcal{P}(n)$. We will consider such generalizations where the property $\mathcal{P}(n)$ is a specified intersection behavior of some subset of the diagonals to a convex n-gon.

We will say that a set $X \subset \mathbb{R}^2$ *generates* a collection $\mathcal{S}$ of segments if each segment in $\mathcal{S}$ has its endpoints in X. Also, we will say that a collection $\mathcal{S}$ of segments is a *crossing family* if each pair of segments in $\mathcal{S}$ crosses (intersects at a point that is not endpoint to either segment). Now if $n \geq 2k$ then the vertex set of a convex n-gon clearly generates a crossing family of size k. Define $c(k)$ to be the least integer such that any planar set of $c(k)$ points in general position generates a crossing family of k segments. Then as we have noted, $c(k) \leq g(2k)$, but we might expect $c(k)$ to be much less. Indeed, the main result in the paper by Aronov el al [1] implies the much stronger bound

$$(0.2) \qquad c(k) \leq 12k^2$$

for these numbers. The authors of that paper ask whether this bound might be improved to a linear bound – a question that remains open at present.

Now let j and k be positive integers. Define a (j, k)-grid to be a collection of segments $\{s_1, s_2, \ldots, s_j, t_1, t_2, \ldots t_k\}$ such that each segment s_i crosses each segment t_j but such that segments s_i and s_j are disjoint if $i \neq j$, as are t_i and t_j. If $n \geq 2j + 2k$ then the vertex set of a convex n-gon clearly generates a (j, k)-grid. Define $\#(j, k)$ be the least integer such that any planar set of $\#(j, k)$ points in general position generates a (j, k)-grid. We again have the easy inequality $\#(j, k) \leq g(2j + 2k)$, but a result of Nielsen and Sabo [5] implies the linear upper bound

$$(0.3) \qquad \#(j, k) < 20k \quad \text{(when } j \leq k)$$

for these numbers.

It appears at least superficially, then, that grids are easier to find than are crossing families, and that both are easier to find than are convex polygons. But while the progression from $g(n)$ to $c(k)$ to $\#(j, k)$ represents a geometric and computational simplification, none of these can be said to be simple. A look at the six-point cases illustrates the situation well.

(A) The value of $g(6)$ is not known. It is conjectured that $g(6) = 17$, but **(0.1)** gives only $17 \leq g(6) \leq 37$.

(B) The value of $c(3)$ is not known, but we will prove in this paper that $9 \leq c(3) \leq 16$.

(C) We will show below that $\#(1, 2) = 8$. However, this is the largest case for which the exact value of $\#(j, k)$ is known.

The purpose of this paper is to establish the facts mentioned in items **(B)** (see Section 1) and **(C)** (see Section 2). Bounds for some of the larger cases of these numbers will be given in a subsequent paper. The methods we use are not complicated, but some imagination is required to find an approach that reduces the number of cases to a manageable level.

1. An improved bound for $c(3)$

The bound **(0.2)** gives only $c(3) \leq 108$ – of course $c(3) \leq g(6) \leq 37$ (from **(0.1)**) is better. We are able here to improve this substantially to $9 \leq c(3) \leq 16$. We would be quite surprised if the actual value of $c(3)$ is not closer to the lower bound, but reducing the upper bound appears to be very difficult.

We begin by developing some notation that will be useful in the main proof. Let X be a finite planar set in general position. Now let A and B be vertices of the convex hull of X admitting parallel supporting lines. We may assume these supporting lines touch the convex hull of X only at points A and B so that the points of $X \setminus \{A, B\}$ lie in the strip between them. One of the half-strips bounded by these lines and the segment AB contains at least half the points of $X \setminus \{A, B\}$. Let this half-strip be called Σ, and let m be the number of points of $X \setminus \{A, B\}$ in Σ. We define sequences of sets $\{X_i\}, \{F_i\}, \{V_i\},$ and $\{W_i\}$ $(0 \leq i \leq m)$ as follows (see Figure 1):

- Let $X_0 = X$.

- For each i such that X_i is defined let F_i be the set consisting of those points in Σ lying on the boundary of the convex hull of X_i, together with the points A and B. Furthermore, set $V_i = X_i \setminus F_i$ and $W_i = X \setminus X_i$.

- Finally, for $i < m$ let C_i be a point of $F_i \setminus \{A, B\}$ and define X_{i+1} to be $X_i \setminus \{C_i\}$.

Note then that $|V_i| + |F_i| + |W_i| = |X|$ and that $W_i = \{C_0, C_1, C_2, \ldots, C_{i-1}\}$, so $|W_i| = i$. We think of F_i as being a "convex fence" separating V_i and W_i.

More generally, we will say that a sequence of points $C_1, C_2, \ldots, C_f$ from X is a (v, w)-*fence* for X if

- $X \setminus \{C_1, C_2, \ldots, C_f\} = V \cup W$ where $|V| = v$ and $|W| = w$,

- $C_1, \ldots, C_f$ are consecutive vertices on the convex hull of $V \cup \{C_1, \ldots, C_f\}$, and

- every segment joining a point of V to a point of W crosses one of the $f - 1$ segments $C_1C_2, C_2C_3, \ldots, C_{f-1}C_f$.
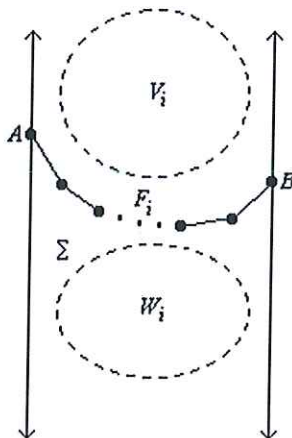
Figure 1.

In this case we say that V is the set of points of X *inside* the fence $C_1 C_2 \cdots C_f$ and W is the of points of X *outside* that fence.

Given positive integers v, f, and w we will say that X has property $\langle v \, / \, f \, / \, w \rangle$ if X has a (v', w')-fence consisting of f points for some $v' \geq v$ and $w' \geq w$. The sets described above yield a sequence of properties $\langle v_0 \, / \, f_0 \, / \, 0 \rangle$, $\langle v_1 \, / \, f_1 \, / \, 1 \rangle$, $\langle v_2 \, / \, f_2 \, / \, 2 \rangle$, ..., $\langle v_m \, / \, f_m \, / \, m \rangle$ for X (where, of course, $v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_m$ and $f_m = 2$).

Lemma 1.1. *Let $X = \{P_1, P_2, P_3, Q_1, Q_2, Q_3, A, B\}$ where for each i and j the segment $P_i Q_j$ crosses AB. Then X generates a crossing family of three segments including AB.*

Proof. Clearly it is enough to show that $P_i P_j Q_k Q_m$ is a convex quadrilateral for some i, j, k, m, and to do this it is enough to show that line $\overleftrightarrow{P_i P_j}$ misses segment $Q_k Q_m$ and line $\overleftrightarrow{Q_k Q_m}$ misses segment $P_i P_j$.

Note that the halfplane determined by $\overleftrightarrow{AB}$ and containing $\{Q_1, Q_2, Q_3\}$ is divided into four (or fewer) regions by the lines $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_2 P_3}$, and $\overleftrightarrow{P_1 P_3}$. Suppose first that one of these regions contains both Q_k and Q_m. Now $\overleftrightarrow{Q_k Q_m}$ cannot intersect all three sides of the triangle $P_1 P_2 P_3$, so it misses some segment $P_i P_j$. From our observation above the points generate a crossing family of six segments.

If none of these regions contains two of the points of $\{Q_1, Q_2, Q_3\}$ then it must be the case that there is a segment $Q_k Q_m$ meeting exactly two of the lines $\overleftrightarrow{P_1 P_2}$, $\overleftrightarrow{P_2 P_3}$, $\overleftrightarrow{P_1 P_3}$. But then $\overleftrightarrow{Q_k Q_m}$ cannot meet any of the sides of the triangle $P_1 P_2 P_3$ (since it cannot meet only one side of the triangle

and there are two triangle sides that it clearly cannot meet, having crossed the lines determined by these sides at points outside of the triangle). But now Q_kQ_m misses one of the lines $\overleftrightarrow{P_iP_j}$, so we once again have the situation described above. $\qquad\square$

Lemma 1.2. *If a set X has property $\langle\, v\,/\,f\,/\,w\,\rangle$ (with $v, w \geq 2$ and $f \geq 3$) and generates no crossing family of three segments then X also has property $\langle\, v - 2\,/\,f - 1\,/\,w - 2\,\rangle$.*

Proof. Let $C_1, C_2, \ldots, C_f$ be a (v, w)-fence for X with related components V and W. Order the points of $V \cup \{C_1, \ldots, C_f\}$ radially from C_2 as $\{P_0 = C_1, P_1, P_2, \ldots, P_{v+f} = C_3\}$ (see Figure 2).
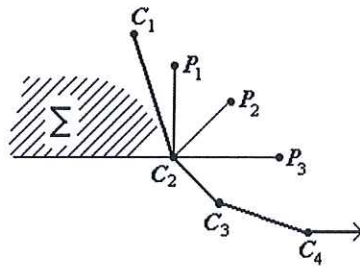
Figure 2.

Now X generates no crossing family of three segments by assumption, so by Lemma 1.1 there cannot be three points of W on the same side Σ of line $\overleftrightarrow{C_2P_3}$ as point C_1. (Again see Figure 2 – the segment C_1C_2 is crossed by every segment joining such a point of W to any one of P_1, P_2, or P_3.) But then $C_2, C_3, \ldots, C_f$ is a fence with related components $V \setminus \{P_1, P_2\}$ and $W \setminus \Sigma$. $\qquad\square$

Lemma 1.3. *Any set having property $\langle\, 2f - 1\,/\,f\,/\,2f - 1\,\rangle$ for some $f \geq 2$ generates a crossing family of three segments.*

Proof. Assume to reach a contradiction that a set X has property $\langle\, 2f - 1\,/\,f\,/\,2f - 1\,\rangle$ for some $f \geq 2$ but generates no crossing family of three segments. Then by Lemma 1.2 X also has property $\langle\, 3\,/\,2\,/\,3\,\rangle$. But by Lemma 1.1 any set with property $\langle\, 3\,/\,2\,/\,3\,\rangle$ *will* generate a crossing family of three segments – a contradiction. $\qquad\square$

These lemmas establish that a set generates a crossing family of three segments if it contains a subset with a $(3, 3)$-fence of two points or a $(5, 5)$-fence of three points. This fact will be used repeatedly in our next proof.

Theorem 1.4. $9 \leq c(3) \leq 16$.

Proof. The lower bound is established by considering the set of eight points depicted in Figure 3. (A few lines are shown to indicate relative positioning of the points.) We leave it to the reader to verify that this set fails to generate any crossing family of three segments.
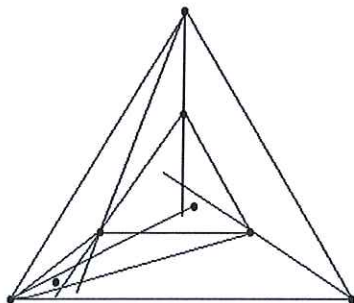

Figure 3.

Now let X be any set of 16 points in general position. We will show that X generates a crossing family of three segments. The construction given at the outset of this section established a sequence $\langle 16 - f_0 / f_0 / 0 \rangle$, $\langle 15 - f_1 / f_1 / 1 \rangle$, $\langle 14 - f_2 / f_2 / 2 \rangle$, ..., $\langle 16 - m - f_m / f_m / m \rangle$ of properties for X. Consider the property $\langle 11 - f_5 / f_5 / 5 \rangle$.

We may clearly assume $f_5 \leq 5$ else X contains the vertices of a convex hexagon and thus clearly generates the desired crossing family. If $f_5 \leq 3$ then we have either $\langle 8 / 2 / 5 \rangle \implies \langle 3 / 2 / 3 \rangle$ or $\langle 7 / 3 / 5 \rangle \implies \langle 5 / 3 / 5 \rangle$, either of which guarantees the crossing family by our preliminary lemmas. It remains, then, to examine the cases $f_5 = 4$ and $f_5 = 5$.

Case 1: $f_5 = 4$

In this case, X has property $\langle 7 / 4 / 5 \rangle$. As in Figure 4, let the fence be $ABCD$ and consider the four regions R_1, R_2, R_3, and R_4 as shown. Let r_i denote the number of points of X in region R_i, so that $r_1 + r_2 + r_3 + r_4 = 7$.
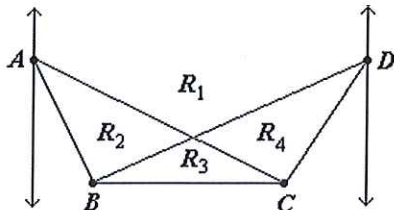

Figure 4.

If $r_1 > 0$ and $r_3 > 0$ (or similarly if $r_2 > 0$ and $r_4 > 0$) it is easy to see that X then generates a crossing family of three segments (two of which are AC and BD). So, if $r_1 > 0$ then we may assume either $r_3 = r_2 = 0$ or $r_3 = r_4 = 0$. This means either ACD or ABD is a $(7, 5)$-fence, so our preliminary lemmas would guarantee the desired crossing family for X. Thus, we may assume $r_1 = r_2 = 0$ and $r_3 + r_4 = 7$.

If $2 \leq r_4 \leq 5$ then (using the points in R_3 and R_4 along with points B and D) AC is a $(3, 3)$-fence. On the other hand, if $r_4 \geq 6$ then ACD is a $(5, 5)$-fence. So (again using our lemmas) we may assume $r_4 \leq 1$.
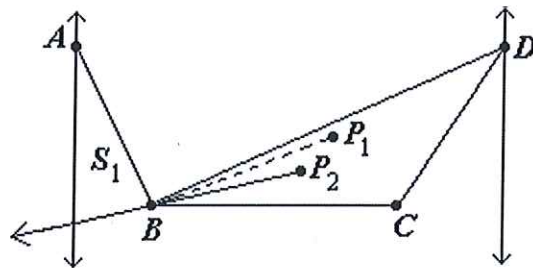
Figure 5.

Subcase 1A: $r_3 = 6$, $r_4 = 1$

Order the points of X in regions R_3 and R_4 as $P_1, P_2, \ldots, P_7$ radially from B as in Figure 5. We may assume that the region S_1 in that figure (bounded by $\overleftrightarrow{AB}$, $\overleftrightarrow{P_2B}$, and the supporting line through A to the convex hull of X) contains at most two points of X, else AB is a $(3, 3)$-fence (with the latter set of three points being $\{P_1, P_2, D\}$). But then BCA is a $(5, 5)$-fence where the points inside consist of the points in region R_3 less P_1 (if $P_1 \in R_3$) and the points outside consist of the (at least three) points outside $ABCD$ not lying in S_1 together with D and the point of X in region R_4. By our preliminary lemmas, X then generates a crossing family of three segments.
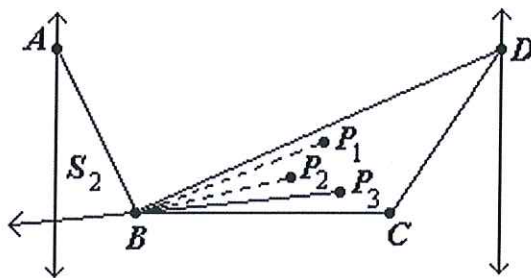

Figure 6.

Subcase 1B: $r_3 = 7$, $r_4 = 0$

Consider now the region S_2 in Figure 6 (bounded by $\overleftrightarrow{AB}$, $\overleftrightarrow{P_3B}$, and the supporting line). If this region contains two points of X then BD is a $(3,3)$-fence where one set of three points is $\{P_1, P_2, P_3\}$ and the other consists of the two points in S_2 together with point A. Thus, we may assume S_2 contains no more than one point of X.

This shows that by discarding up to two points (P_1 and P_2) inside and one point (in S_2) outside the fence $ABCD$ we can eliminate all segments (joining an inside point to an outside point) that cross AB. A mirror image of this same argument can be done for eliminating segments that cross CD. In this way we conclude that we can discard four points inside $ABCD$ and two points outside $ABCD$ and eliminate all segments except those that cross BC. So, BC is a $(3,3)$-fence for the remaining subset of X. As before, this is sufficient to guarantee that X generates the desired crossing family.

Case 2: $f_5 = 5$

In this case, X has property $\langle\, 6\,/\,5\,/\,5\,\rangle$. Consider the convex hull of the five points making up the fence $ABCDE$. The diagonals of this pentagon determine eleven interior regions. If a point of X lies in any of the five regions bounded by segments of the pentagon or in the central region bounded by all the diagonals, then X generates a crossing family of three segments as shown in Figure 7.
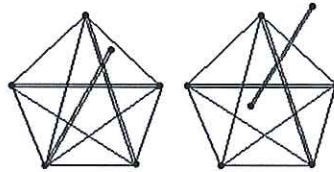

Figure 7.

Thus, we may assume that the six points of X inside the fence $ABCDE$ lie in the five regions labeled R_1 through R_5 in Figure 8. As before, let r_i denote the number of points of X in region R_i.
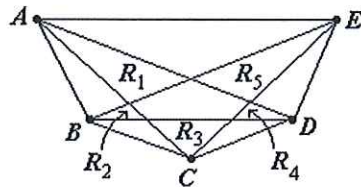

Figure 8.

- If points of X lie in each of two adjacent regions from among R_1 through R_5 then it is easy to construct a crossing family of three segments (two are diagonals of $ABCDE$ and the other joins the points in question).

- If $r_2 = r_4 = 0$ then ACE is a $(6,5)$-fence – more than enough to guarantee the desired crossing family.

Putting together these two observations, we may now assume that $r_2 > 0$, $r_1 = r_3 = 0$, either $r_4 = 0$ or $r_5 = 0$, and (of course) $r_2 + r_4 + r_5 = 6$.

- If $2 \le r_2 \le 5$ then AC is a $(3,3)$-fence where on one side we include two of the points from region R_2 along with B and on the other side we include a point from $R_4 \cup R_5$ along with D and E.

- If $r_2 = 1$ then either R_4 or R_5 contains 5 points of X. If R_4 contains these points then CE is a $(3,5)$-fence where the three points on one side consist of the point in R_2 together with A and B. If the points lie in R_5 then similarly AD is a $(3,5)$-fence.

Both of the above cases, then, lead to a crossing family of three segments generated by X. The only remaining case to consider is $r_2 = 6$ (and $r_1 = r_3 = r_4 = r_5 = 0$). Here, $ABDE$ is a $(6,6)$-fence for X (where we include C with the points outside the fence). Note that Lemma 1.2 would allow us to conclude that either X generates a crossing family of three segments or else its property $\langle 6/4/6 \rangle$ must imply property $\langle 4/3/4 \rangle$ (and $\langle 2/2/2 \rangle$). Unfortunately, that is not sufficient under our lemmas to give the desired conclusion. Instead, we will need to be more careful in reducing the fence.

For our first step in the reduction, order the points of X in region R_2 as P_1 through P_6 radially from B as in Figure 9. We may assume the region S_1 in that figure contains no more than one point of X, since otherwise BE is a $(3,3)$-fence (with two points from S_1 and A on one side and $\{P_1, P_2, P_3\}$ on the other). Then discarding any point in S_1 along with P_1 and P_2, we may eliminate all segments (joining a point inside $ABDE$ to a point outside) that cross AB; all this while leaving at least four inside points and at least five outside points.
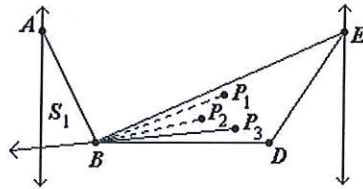

Figure 9.

The second part of the reduction is accomplished by ordering the remaining points of X in R_2 as Q_1 through Q_4 radially from D as in Figure 10. We may assume the region labeled as S_2 in that figure contains no more than 2 points of X, else DE is a $(3,3)$-fence (with three points from S_2 on one side and $\{A, Q_1, Q_2\}$ on the other). Thus, discarding points in S_2 along with Q_1, BD is now a $(3,3)$-fence for the remaining set, and our reduction is complete.
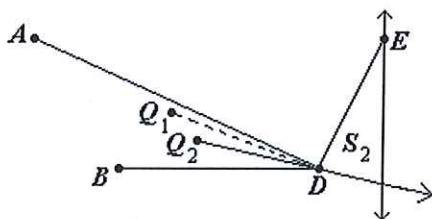

Figure 10.

We have demonstrated that in every possible case the set X must generate a crossing family of three segments, so the theorem is proved. $\square$

2. An exact value for $\#(1,2)$

Here we prove the only exact value known for any of the six-point configuration numbers mentioned in the introduction. The following well-known fact will prove useful in the analysis.

Lemma 2.1: *Suppose that $X = \{A, B, P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n\}$ where for each i and j the segment P_iQ_j crosses AB. Then X generates a $(1,n)$-grid consisting of n pairwise disjoint segments each of which crosses AB.*

Proof. Let π be the permutation of $\{1, 2, \ldots, n\}$ such that the sum of the lengths of the segments $P_nQ_{\pi(n)}$ is minimized. Then the segments $\{AB, P_1Q_{\pi(1)}, P_2Q_{\pi(2)}, \ldots, P_nQ_{\pi(n)}\}$ form a $(1,n)$-grid. $\square$

Theorem 2.2: $\#(1,2) = 8$.

Proof. Figure 11 shows a set of seven points that generates no $(1,2)$-grids. (This is easily checked by hand.)

It remains to prove that any set of eight points in general position will generate a $(1,2)$-grid. Let X be any such set, let Z be a vertex of the convex hull of X, and let $X' = X \setminus \{Z\}$. We consider several cases depending on the shape of the convex hull of X'.
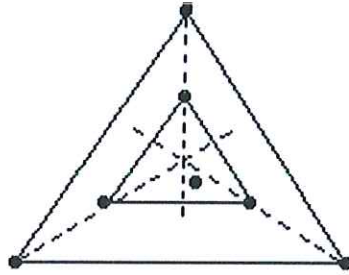
Figure 11.

Case 1. If the convex hull of X' is a 6-gon or 7-gon then X' clearly generates a $(1, 2)$-grid.

Case 2. Suppose the convex hull of X' is a 5-gon $ABCDE$ with the remaining points of X' being P and Q. If P and Q lie on opposite sides of any diagonal to $ABCDE$ then X' generates a $(1,2)$-grid by Lemma 2.1. Consequently, we may assume that P and Q lie in the "inner pentagon" determined by the diagonals. We may also assume that Q is interior to the triangle PAB. But then QD crosses both the diagonal CE and either PA or PB, giving us a $(1,2)$-grid (see Figure 12).
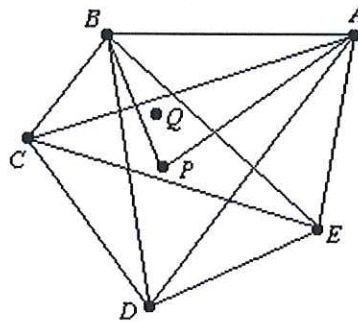

Figure 12.

Case 3. If the convex hull of X' is a quadrilateral $ABCD$ then the three points $\{P, Q, R\} = X' \setminus \{A, B, C, D\}$ must be distributed between the four regions determined by the diagonals of $ABCD$. By Lemma 2.1 we may assume that these points are not separated by either diagonal – thus all lie in the same region.

Assume then that P, Q, and R are each interior to both triangles ABC and BCD. If points $\{A, P, Q, R\}$ form the vertices of a convex quadrilateral then its diagonals together with segment BD form a $(1, 2)$-grid (see the left half of Figure 13). Thus, we may also assume that R is interior to the

triangle APQ as in the right half of Figure 13. We now consider some subcases. Recall that there is a point Z of set X lying outside of the convex hull of X'. The segment RZ must meet either BC or one or both of the diagonals of $ABCD$
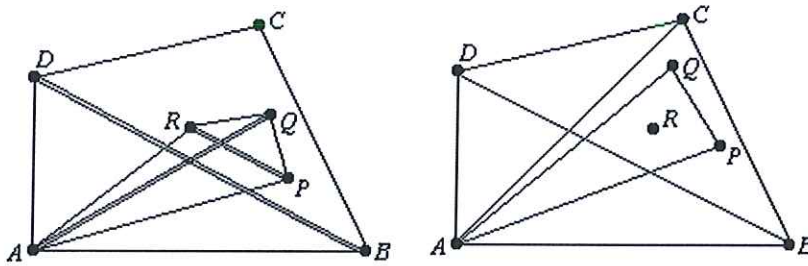

Figure 13.

Subcase 3A. Suppose first that RZ meets BC. Note that it must also meet one of the sides of triangle APQ, and that this side is disjoint from BC. Thus X generates a (1,2)-grid in this case.

Subcase 3B. Next suppose that RZ meets exactly one of the diagonals of $ABCD$. Then these diagonals together with RZ form a (1,2)-grid.

Subcase 3C. Finally, suppose that RZ meets both diagonals of $ABCD$. In particular, then, RZ meets BD. But BD meets both AP and AQ, and RZ must miss one of these. So, this case also yields a generated (1,2)-grid.

Case 4. The only remaining case is to assume that the convex hull of X' is a triangle, say ABC, with the four remaining points of X' interior to this triangle. If $ZABC$ is a convex quadrilateral then by separating B from the remaining seven points we reduce to one of the earlier cases (see Figure 14).
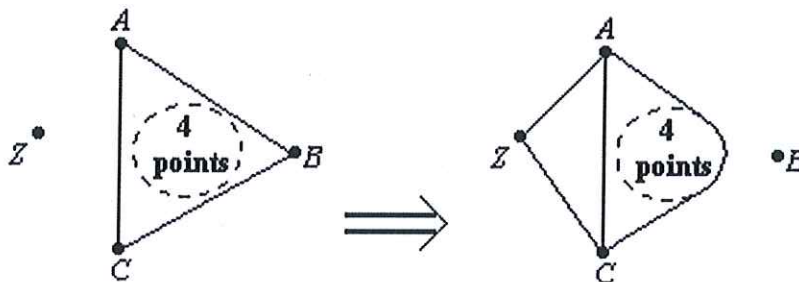

Figure 14.

Thus we may assume that C is interior to triangle ZAB (so that the convex hull of X is a triangle). We may clearly assume that a similar configuration results if any of the three vertices of this triangle are separated from the other seven points. In this case the set X must be as pictured in Figure 15: the convex hull of X is a triangle $Z_1 Z_2 Z_3$ and the convex hull of $X \setminus \{Z_i\}$ is a triangle with C_i as the third vertex. Two points of X, say P and Q, must lie in the region common to the interiors of triangles $C_1 Z_2 Z_3$, $C_2 Z_1 Z_3$, and $C_3 Z_1 Z_2$. We consider two subcases for the placement of these points.
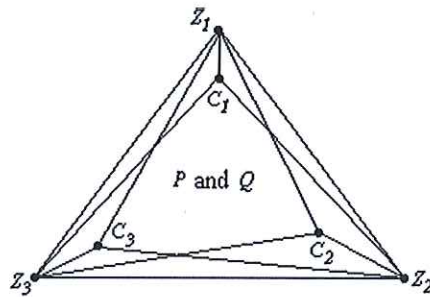

Figure 15.

Subcase 4A. First it is possible that either P or Q is not interior to triangle $C_1 C_2 C_3$. Assume, for instance, that segment $C_2 C_3$ separates P from Z_1 as in Figure 16. In this case PZ_1 meets both $C_2 C_3$ and one of $C_1 Z_2$ or $C_1 Z_3$, yielding a (1,2)-grid.
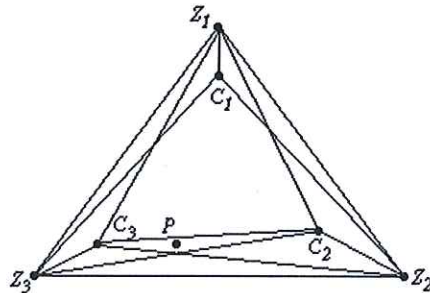

Figure 16.

Subcase 4B. Finally, assume both P and Q lie interior to triangle $C_1 C_2 C_3$. The ray $\overrightarrow{PQ}$ must meet one of the sides of this triangle, say $C_2 C_3$. Then Q is interior to triangle $PC_2 C_3$ so we have the configuration depicted in Figure 17. The segment QZ_1 must now meet a side of each of the triangles $C_1 Z_2 Z_3$ and $PC_2 C_3$, again yielding a (1,2)-grid.
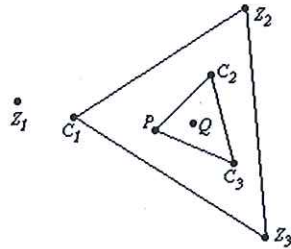
Figure 17.

All cases have now been considered and the proof is complete. □

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