Crossing Fans of Segments Determined by a Finite Point Set

Mark J. Nielsen
Brenna Peever
Department of Mathematics
University of Idaho
Moscow, ID 83844-1103

Abstract. Let S be a finite set of points in the plane in general position. We investigate the number of points S must contain in order to guarantee the existence of specified patterns of intersecting segments determined by S. Specifically we are interested in collections of "fans" of segments sharing a single point of S in which each pair of segments from distinct fans cross.

Let S be a finite set in general position in the plane (no three points on a line) and let $\{p_0, p_1, p_2, \ldots, p_k\}$ be a subset of $k+1$ distinct points in S. We then call the collection of segments $\mathcal{F} = \{p_0p_1, p_0p_2, \ldots, p_0p_k\}$ a *k-fan* determined by S. We say that segment q_0q_1 *crosses* the k-fan $\mathcal{F}$ if q_0q_1 intersects every segment in $\mathcal{F}$ at a point not in S. If $\mathcal{F}_1$ is a k-fan and $\mathcal{F}_2$ is a j-fan so that every segment of $\mathcal{F}_2$ crosses $\mathcal{F}_1$, then we refer to the pair $(\mathcal{F}_1, \mathcal{F}_2)$ as a *(k, j)-pair of crossing fans.* We refer to a $(k, 1)$-pair of crossing fans as simply a *crossed k-fan.* If $\mathcal{F}_i$ is a k_i-fan for $1 \le i \le s$ so that each pair $(\mathcal{F}_i, \mathcal{F}_{i'})$ $(1 \le i < i' \le s)$ is a crossing pair, then we say that $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_s)$ is a $(k_1, k_2, \ldots, k_s)$-family of crossing fans.

In this paper we will investigate the number of points required to guarantee that S will determine these patterns of crossing segments. In particular, for positive integers $k_1, k_2, \ldots, k_s$ we define $F(k_1, k_2, \ldots, k_s)$ to be the least integer n such that any set of n points in general position in the plane determines a $(k_1, k_2, \ldots, k_s)$-family of crossing fans. As a special case, note that $F(1, k)$ is simply the number of points needed to guarantee that S determines a crossed k-fan.

Our motivation for these definitions comes from a very old problem: for each positive integer $n \ge 3$ let $g(n)$ be the least integer such that every planar set of $g(n)$ points in general position contains the vertices of some convex n-gon. Erdös and Szekeres introduced this problem in 1935 (see [2] and [3]), and despite much study in subsequent years, exact values for g are known only for $n \le 5$ along with the bound

$$2^{n-2} + 1 \le g(n) \le \binom{2n-5}{n-2} + 2\,.$$

(see [1], [4], and [8]). Morris and Soltan [5] provide an excellent survey of related results. The segment patters we seek here are all easily found among the diagonals of a convex m-gon for large enough m. In fact, it is easy to see that $F(k_1, k_2, \ldots, k_s) \leq g(m)$ where $m = s + \sum_{i=1}^{s} k_i$. Questions about $F(k_1, k_2, \ldots, k_s)$, then, may be viewed as generalized weakenings of the Erdös and Szekeres problem. Similar generalizations were investiaged in papers by Nielsen and Sabo [6] and Nielsen and Webb [7]. In this paper we establish the following facts:

Theorem 1. *$F(1, k) \leq 2k + 3$ for every k, and equality holds for $k \leq 3$.*

Theorem 2. $F(2, 2) \leq 9$.

Theorem 3. $F(j,k) \leq \begin{cases} 4j^2 + 3jk - 6j + 4 \text{ if } 2 \leq j \leq k \leq 2j - 2 \\ 6j^2 + 2jk - 6j + 4 \text{ if } 2 \leq j \text{ and } k \geq 2j - 1 \end{cases}$

1. Proving Theorem 1.

We will denote the convex hull of set S by $\text{conv}(S)$. The following lemma is key to our first theorem.

Lemma A. *If S is a set of $2k + 2$ points in general position and* $\text{conv}(S)$ *is not a triangle then S determines a crossed k-fan.*

Proof. Let S be as stated in the lemma and let p_1, p_2, and p_3 be consecutive vertices of $\text{conv}(S)$. Let m be the number of points of S on the side of line $\overleftrightarrow{p_1p_3}$ opposite p_2 (see the leftmost part of Figure 1). Note that by assumption we have $m \geq 1$. Also, the number of points of S interior to the triangle $p_1p_2p_3$ must be $2k - m - 1$.

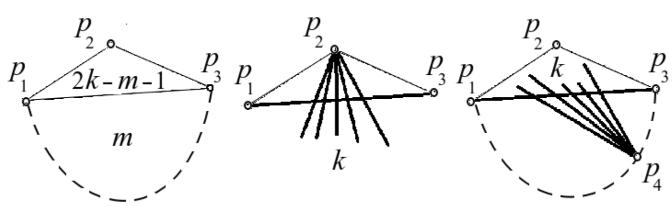

Figure 1.

Now clearly if $m \geq k$ then S determines a k-fan (with segments sharing p_2) crossed by segment p_1p_3 (see the middle portion of Figure 1). But if $m < k$ then $2k - m - 1 \geq k$. So if v_4 is a vertex of $\text{conv}(S)$ on the opposite side of $\overleftrightarrow{p_1p_3}$ from p_2, then S generates a k-fan of segments sharing v_4 that is crossed by segment p_1p_3 (see the rightmost part of Figure 1). $\square$

Proof of Theorem 1: Let S be a set of $2k+3$ points in general position. If $\mathrm{conv}(S)$ is not a triangle then S determines a crossed k-fan by Lemma A, so assume $\mathrm{conv}(S)$ is a triangle with vertices p_0, q_1, and q_2. Now $S \setminus \{p_0\}$ is a set of $2k+2$ points, so again by Lemma A we may assume that $\mathrm{conv}(S \setminus \{p_0\})$ is also a triangle. Let the vertices of $\mathrm{conv}(S \setminus \{p_0\})$ be p_1, q_1, and q_2. Then since there are $2k-1$ points of S interior to $\mathrm{conv}(S \setminus \{p_0\})$, k of these points must lie on one side of line $\overleftrightarrow{p_0 p_1}$, and thus S determines a crossed k-fan as in Figure 2.

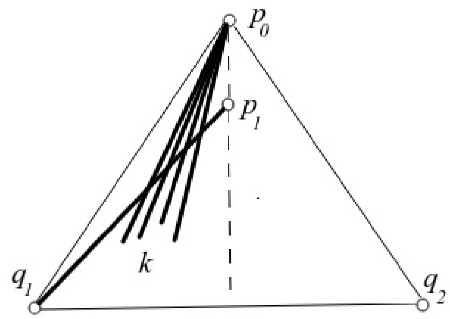

Figure 2.

This establishes that $F(1,k) \leq 2k+3$. The fact $F(1,1) = 5$ is trivial. We leave it to the reader to verify that the set of 8 points depicted in Figure 3 shows that $F(1,3) = 9$. Removing the innermost two points from this set yields an example showing $F(1,2) = 7$. $\square$

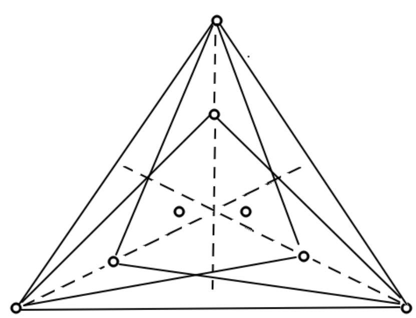

Figure 3.

2. Proving Theorem 2. The proof for Theorem 2 is accomplished by establishing preliminary cases in a lemma.

Lemma B. *Each of the following determines a (2,2)-pair of crossed fans:*

(i) *A set of six points in convex position.*

(ii) *A set of seven points whose convex hull is a pentagon.*

(iii) *A set of seven points whose convex hull is a quadrilateral.*

Proof. Part **(i)** is immediate. For part **(ii)** let S be a set of seven points with a pentagonal convex hull. Let p and q be the points of S interior to the convex hull. The left half of Figure 4 shows that if either p or q is in a triangular region created by one of the hull's diagonals, then S determines a $(2, 2)$-pair of crossing fans. Thus, we may assume that both p and q lie in the interior pentagonal region formed by the diagonals, as in the right half of Figure 4. But S then determines a $(2, 2)$-pair of crossing fans as shown in that figure.

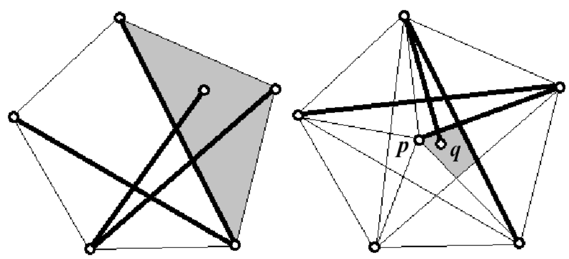

Figure 4.

It remains only to demonstrate **(iii)**. Let S be a set of seven points with convex hull the quadrilateral $v_1v_2v_3v_4$ and with points $\{p_1, p_2, p_3\} \subset S$ interior to that quadrilateral. Now the diagonals v_1v_3 and v_2v_4 divide the interior of the hull into four triangular regions.

Case (iii-a) The left diagram in Figure 5 shows that if $\{p_1, p_2, p_3\}$ intersects two of these regions that are not adjacent, then S determines a $(2, 2)$-pair of crossing fans. So we may without loss of generality assume that $\{p_1, p_2, p_3\}$ lies entirely on one side of one of the diagonals, say the same side of diagonal v_1v_3 as v_2.

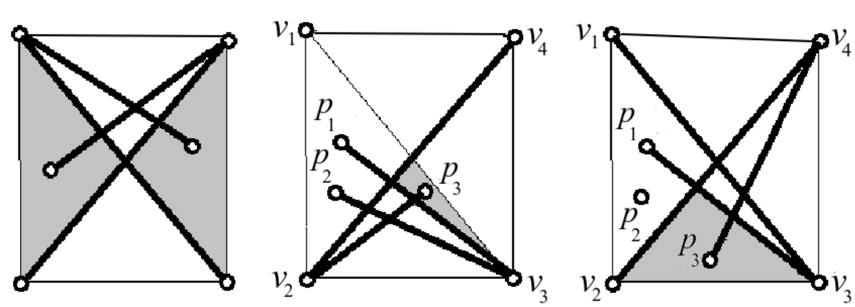

Figure 5.

Case (iii-b) Suppose that $\{p_1, p_2, p_3\}$ is not contained in any one of the triangular regions determined by the diagonals. In particular, we may assume without loss of generality that p_1 and p_2 lie in the triangular region touching side v_1v_2 while p_3 lies in the region touching side v_2v_3. Viewing the points of S radially from v_3, the points p_1, p_2, and p_3 will occur in some order between v_1 and v_2. If p_3 appears first in this order, the middle diagram of Figure 5 demonstrates that S determines a $(2,2)$-pair of crossing fans. But if p_3 is *not* first in this order then the right diagram in Figure 5 likewise shows the existence of a $(2,2)$-pair of crossing fans.

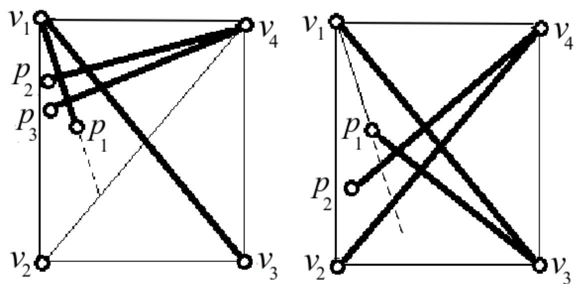

Figure 6.

Case (iii-c) With the two above cases eliminated, we may now assume that $\{p_1, p_2, p_3\}$ lies entirely in one of the triangular regions, say the one touching side v_1v_2. We may further assume that the radial order of points viewed from v_1 is v_3, p_1, p_2, p_3, v_2. If both segments p_2v_4 and p_3v_4 intersect p_1v_1 then the diagram on the left half of Figure 6 shows a determined $(2,2)$-pair of crossing fans. Otherwise (if, say p_2v_4 misses p_1v_1) the right half of that same figure demonstrates the existence of the desired pair of fans. $\square$

Proof of Theorem 2: Let S be a set of nine points in general position. From Lemma B we see that S determines a $(2,2)$-pair of crossing fans so long as $\text{conv}(S)$ is not a triangle. So, suppose that $\text{conv}(S)$ is a triangle $p_1q_1r_1$. We may likewise apply Lemma B to the eight point set $S \setminus \{p_1\}$ and reduce to the case that $\text{conv}(S \setminus \{p_1\})$ is a triangle $p_2q_1r_1$. Applying Lemma B once more to the seven point set $S \setminus \{p_1, p_2\}$, we may assume that $\text{conv}(S \setminus \{p_1, p_2\})$ is a triangle $p_3q_1r_1$.

We could, of course, apply the same reasoning to conclude that $\text{conv}(S \setminus \{q_1\})$ must be a triangle $q_2r_1p_1$, and that $\text{conv}(S \setminus \{q_1, q_2\})$ must be a triangle $q_3r_1p_1$. Then S determines a $(2,2)$-pair of crossing fans as shown in Figure 7. $\square$

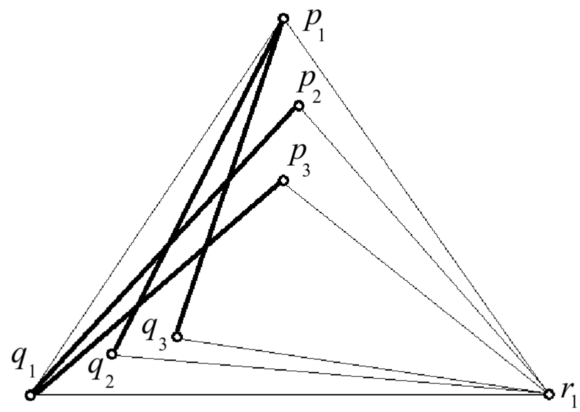

Figure 7.

While we do not include the proof here, we have been able to use an analysis similar to that above to prove that $F(2,3) \leq 11$.

3. Proving Theorem 3. We again begin with preliminary observations. For the following lemmas, assume that S is a finite set of points in general position and that $\mathrm{conv}(S)$ is the polygon $p_0p_1p_2\ldots p_s$. For each i $(1 \leq i \leq s-1)$ let d_i be the number of points of S on the side of $\overleftrightarrow{p_0p_i}$ opposite p_{i+1} and let d_i' be the number of points of S on the side of $\overleftrightarrow{p_0p_{i+1}}$ opposite p_i.

Lemma C. *Suppose $j \leq k$ that for some i we have $d_i \geq j$ and $d_i' \geq k-1$. Then S determines a (j,k)-pair of crossing fans.*

Proof. The proof is immediate – see Figure 8. □

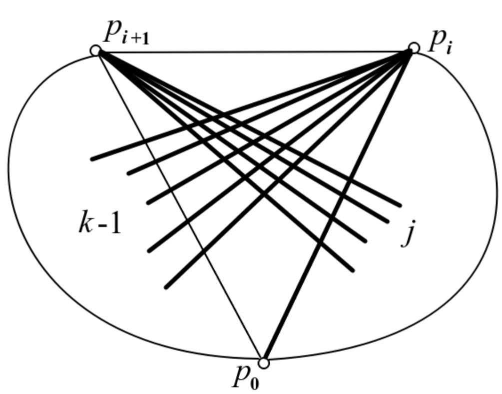

Figure 8.

Lemma D. *Assume that for some i the two sides of $\overleftrightarrow{p_0p_i}$ contain respectively at least $2j-1$ points of S and at least $2k-1$ points of S. Then S determines a (j,k)-pair of crossing fans.*

Proof. Let p_q be a vertex of $\operatorname{conv}(S)$ on the side of $\overleftrightarrow{p_0p_i}$ that contains at least $2k-1$ points of S. Let R be a ray from p_q such that each side of R contains $k-1$ points from S on that side of $\overleftrightarrow{p_0p_i}$. There are two cases, according to whether or not R meets the segment p_0p_i. Both cases lead to a (j,k)-pair of crossing fans, as seen in Figure 9. □

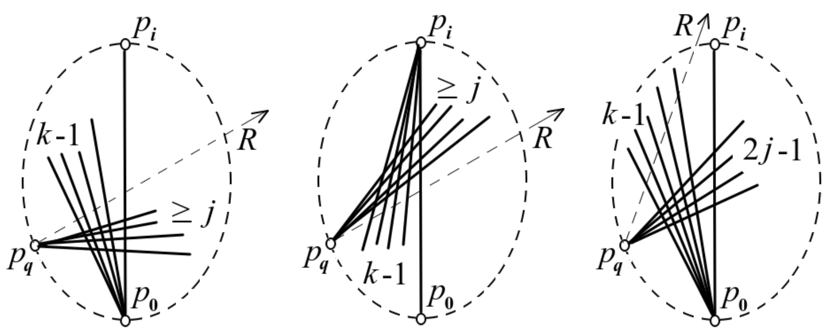

Figure 9.

Our final lemma establishes that if S determines no (j,k)-pair of crossing fans then there is an i for which $d_i + d'_i$ is relatively small. This means that a substantial portion of S is contained in the triangle $p_0p_ip_{i+1}$.

Lemma E. *Suppose $j \leq k$ and $|S| = n \geq 4k-1$. If S does not determine any (j,k)-pair of crossing fans then there is some i so that d_i and d'_i are both no more than $2j-2$, and either one of them is no more than $k-2$ or both of them are no more than $j-1$.*

Proof. Let i be the largest index for which $d_i \leq 2j-2$. (Note that we may assume $i < s-1$ since otherwise $d'_i = 0$ and the lemma holds trivially.) Then $d_{i+1} \geq 2j-1$, so from Lemma D (applied to $\overleftrightarrow{p_0p_{i+1}}$) we see that $d'_i \leq 2k-2$. But this means $d_{i+1} \geq n-2k \geq 2k-1$, so Lemma D implies we must in fact have $d'_i \leq 2j-2$. Finally, from Lemma C we see that at least one of d_i or d'_i must be less than $k-1$ (or in the case that $j=k$ they may both be equal to $j-1$). □

Proof of Theorem 3: We will give the proof for the case that $2 \leq j \leq k \leq 2j-2$. The proof for the case $k \geq 2j-1$ is a simple variation.

Let S be a set of $4j^2 + 3jk - 6j + 4$ points in general position and assume (to reach a contradiction) that S does not determine any (j,k)-pair of crossing fans. Note that since $k \leq 2j-2$ we may take the conclusion of Lemma E to be that $d_i + d'_i \leq 2j+k-4$ for some i. Consider the follwoing construction.

- Begin with $S_0 = S$ and let u_0 be a vertex of $\mathrm{conv}(S_0)$.
- Having defined S_m with u_0 a vertex of its convex hull (and assuming for the moment that $|S_m| \geq 4k-1$, apply Lemma E to S_m to find consecutive vertices v_m and w_m of $\mathrm{conv}(S_m)$ and a set $D_m \subset S_m$ so that
  - u_o, v_m, and w_m appear as vertices of $\mathrm{conv}(S_m)$ in counterclockwise order,
  - $|D_m| \leq 2j+k-4$, and
  - $\mathrm{conv}(S_m \setminus D_m)$ is the triangle $u_0 v_m w_m$.

  Define S_{m+1} to be $(S_m \setminus D_m) \setminus \{v_m\}$ (see Figure 10).

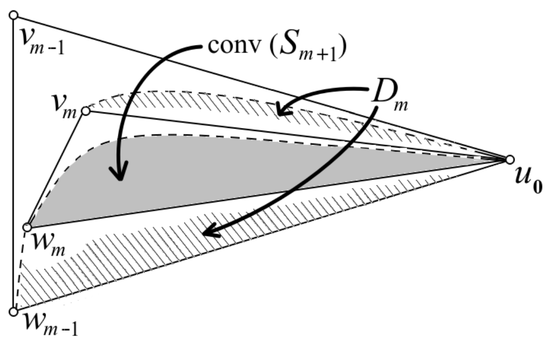

Figure 10.

This yields sets $S = S_0 \supset S_1 \supset S_2 \supset \cdots$ and the construction must terminate when $|S_m|$ falls below $4k-1$ (as required by Lemma E). We will show below that it is possible to extend the construction to S_{2j}. There are three notes about the construction that will be useful in the analysis.

(i) At each stage we have $|S_{m+1}| = |S| - |\{v_0, v_1, \ldots, v_m\}| - \sum_{i=0}^{m} |D_i|$.

(ii) It is possible that $w_m = w_{m-1}$, but if this occurs then $|D_m| \leq 2j-2$ whereas in general we can only say $|D_m| \leq 2j+k-4$.

(iii) At each stage the polygon $u_0 w_0 w_1 w_2 \ldots w_m v_m$ is convex. (As noted in **(ii)**, some of the vertices in this list may be repeated.)

To show the construction of S_{2j} is possible we must show $|S_{2j-1}| \geq 4k-1$. An important part of this verification is the following fact.

Claim: The list $w_0, w_1, w_2, \ldots, w_{2j-1}$ contains no more than j distinct points.

To see this, suppose (to reach a contradiction) that the distinct entries in that list are (in order of increasing index) $w_0, w_{m(1)}, w_{m(2)}, w_{m(3)}, \ldots$ and that this list reaches to $w_{m(j)}$ (with $m(j) \leq 2j-1$). Now $w_{m(j)}$ is found by applying Lemma E to $S_{m(j)}$, and from **(i)** above we have

$$\left|S_{m(j)}\right| = |S| - \left|\{v_0, v_1, \ldots, v_{m(j)-1}\}\right| - \sum_{i=0}^{m(j)-1} |D_i|.$$

But the sum $\displaystyle\sum_{i=0}^{m(j)-1} |D_i|$ includes j entries bounded by $2j+k-4$ ($|D_0|$ and $|D_i|$ for indicies $i \geq 1$ for which $w_i \neq w_{i-1}$) as well as up to $j-1$ additional entries bounded by $2j-2$ (see note **(ii)** above). Thus:

$$\begin{aligned}
\left|S_{m(j)}\right| &\geq |S| - m(j) - [j(2j+k-4) + (j-1)((2j-2)] \\
&\geq (4j^2 + 3jk - 6j + 4) - (2j-1) - (4j^2 + jk - 8j + 2) \\
&\geq 2jk + 3 \\
&> 4k
\end{aligned}$$

so we may indeed apply Lemma E to $S_{m(j)}$. This will result in removing the point $v_{m(j)}$ together with up to $2j+k-4$ points in $D_{m(j)}$ so that

$$\left|S_{m(j)+1}\right| \geq (2jk+3)-1-(2j+k-4) = (2j-1)(k-1)+5 \geq 3(k-1)+5 > 3k.$$

But since $w_0 w_{m(1)} w_{m(2)} \cdots w_{m(j)} v_{m(j)} u_0$ is convex (note **(iii)** above), it is easy to see (Figure 11) that this would result in a k-fan (formed by joining k points in $S_{m(j)+1}$ to w_0) crossing the j-fan formed by joining u_0 to the points $\{w_{m(1)}, w_{m(2)}, \ldots, w_{m(j)}\}$. This contradiction proves our claim.

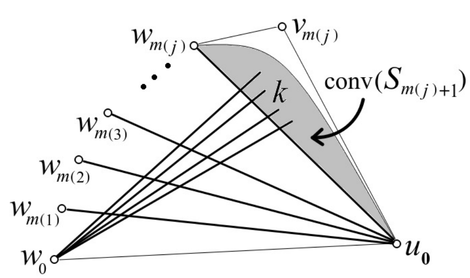

Figure 11.

Now note that $|S_{2j-1}| = |S| - |\{v_0, v_1, \ldots, v_{2j-2}\}| - \sum_{i=0}^{2j-2} |D_i|$ and the same argument as above for $S_{m(j)}$ establishes that $|S_{2j-1}| > 4k$. Thus the construction of S_{2j} is in fact possible.

Once again using fact **(i)**, note that

$$
\begin{aligned}
|S_{2j}| &= |S| - |\{v_0, v_1, \ldots, v_{2j-1}\}| - \sum_{i=0}^{2j-1} |D_i| \\
&\geq (4j^2 + 3jk - 6j + 4) - 2j - [j(2j + k - 4) + j(2j - 2)] \\
&\geq 2j(k-1) + 4
\end{aligned}
$$

This means that there are at least $2j(k-1)+1$ points of S_{2j} in the interior of the triangle $u_0 v_{2j-1} w_{2j-1}$. Since the $2j-1$ rays from v_0 through the points $\{v_1, v_2, \ldots, v_{2j-1}\}$ subdivide the interior of this triangle into at most $2j$ regions, one such region, call it R, must contain at least k points of S.

Finally, observe that we may partition the index set $\{1, 2, 3, \ldots, 2j-1\}$ into disjoint subsets $\mathcal{I}_1 \cup \mathcal{I}_2$ where

$\mathcal{I}_1 = \{i :$ segment pv_0 intersects segment $u_0 v_i$ for every $p \in R\}$ and

$\mathcal{I}_2 = \{i :$ segment pv_0 intersects segment $w_{2j-1} v_i$ for every $p \in R\}$

(see Figure 12). Define $\mathcal{F}_1$ to be the fan formed by joining u_0 to each of the points $\{v_i : i \in \mathcal{I}_1\}$ and $\mathcal{F}_2$ to be the fan formed by joining w_{2j-1} to each of the points $\{v_i : i \in \mathcal{I}_2\}$. Both of these fans cross the k-fan formed by joining k points from $S \cap R$ to v_0, and one of $\mathcal{F}_1$ or $\mathcal{F}_2$ must clearly have size at least j. This contradicts our assumption that S does not admit a (j, k)-pair of crossing fans, and at last completes the proof of Theorem 3.

□

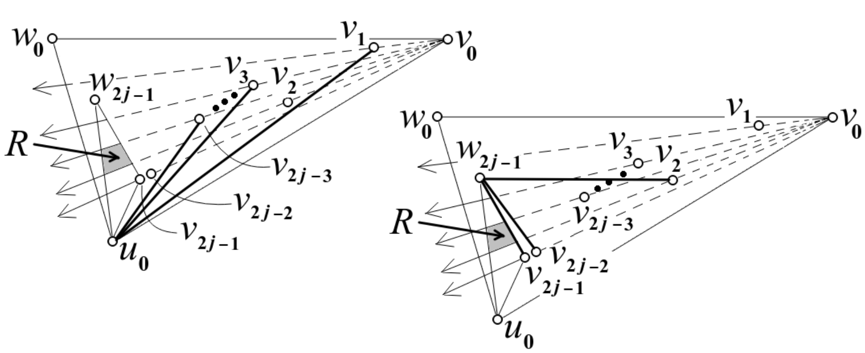

Figure 12.

The bound in Theorem 3 is clearly subject to some improvement, though we suspect that any such bound will be quadratic in j and k.

References

[1] F.R.L. Chung and R.L. Graham, Forced convex n-gons in the plane, *Discrete Comput. Geom.* **19** (1998), pp.367-371.

[2] P. Erdös and G. Szekeres, A combinatorial problem in geometry, *Composito Math.* **2** (1935), pp.463-470. Reprinted in: *Paul Erdös: Selected Writings* (J. Spenceer, ed.), pp.3-12, MIT Press, Cambridge, MA, 1973. Also reprinted in: *Classic Papers in Combinatorics* (I. Gessel and G.-C. Rota eds.), pp.49-56, Birkhäuser, Basel, 1987.

[3] P. Erdös and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **3-4** (1961), pp.53-62. Reprinted in: *Paul Erdös: The Art of Counting. Selected Writings* (J. Spencer, ed.), pp.680-689, MIT Press, Cambridge, MA, 1973.

[4] D. Kleitman and L. Pachter, Finding convex sets among points in the plane, *Discrete Comput. Geom.* **19** (1998), pp.405-410.

[5] W. Morris and V. Soltan, The Erdös-Szekeres problem on points in convex position - a survey, *Bulletin of the A.M.S.* **37** (2000), pp.437-458.

[6] M.J. Nielsen and D.E. Sabo, Transverse families of matchings in the plane, *Ars Combinatoria* **55** (2000), pp.193-199.

[7] M.J. Nielsen and W. Webb, On some numbers related to the Erdös-Szekeres Theorem, *Open Journal of Discrete Mathematics* **3** (2013), pp.167-173.

[8] G. Tóth and P. Valtr, Note on the Erdös-Szekeres theorem, *Discrete Comput. Geom.* **19** (1998), pp.457-459.