Multivariate Normal Distribution Exercise

Given that

\[ X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \sim \mathcal{N}_3(\mu, \Sigma) \]

where

\[ \mu = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \]

and

\[ \Sigma = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \]

Another way to write is \( \{X_1, X_2, X_3\} \sim \mathcal{N}_3(\mu, \Sigma) \)

1. Write down the density of \( f(X) \)
2. Find the correlation matrix \( \rho \) of \( X \)
3. Find the marginal distribution of \( X_2 \)
4. Find the marginal distribution of \( \{X_1, X_3\} \)
5. Find the marginal distribution of \( \{X_1, X_2\} \)
6. Find the conditional distribution of \( X_1|X_3 = -1 \)
7. Find the conditional distribution of \( X_1|\{X_2 = 1, X_3 = -1\} \)
8. Find the conditional distribution of \( \{X_1, X_2\}|X_3 = -1 \)
9. Is \( \{X_1, X_3\} \) and \( X_2 \) independent?
10. Is \( a_1X_1 + a_3X_3 \) and \( a_2X_2 \) independent for all constants \( a_1, a_2, \) and \( a_3? \)
11. Is \( X_1 + X_2 \) and \( X_1 - X_2 \) independent?
12. Let \( Y = AX + a \) where \( A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \) and \( a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \). Find the distribution of \( Y \).
13. Let \( W = (X - \mu)'\Sigma^{-1}(X - \mu) \). What is the distribution, mean, and variance of \( W \)?
14. Find the 95% confidence ellipsoid for \( X \).
15. Let \( X^* = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \). Find the 95% confidence ellipse for \( X^* \).
16. Draw the 95% confidence ellipse for \( X^* \) by finding the eigenvalues and eigenvectors first.
17. Let \( G \) be such that \( GG' = \Sigma^{-1} \). Show that \( G'X \sim \mathcal{N}_3(G'\mu, I) \) and \( G'(X - \mu) \sim \mathcal{N}_3(0, I) \).
   Hint: \( (AB)^{-1} = B^{-1}A^{-1} \)