

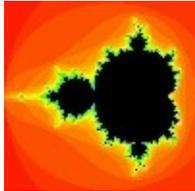


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September 2006

Features



Unveiling the Mandelbrot set

by Robert L. Devaney

Back in the 1970s and 1980s, mathematicians working in an area called dynamical systems made use of the ever-advancing computing power to draw computer images of the objects they were working on. What they saw blew their minds: fractal-like structures whose beauty and complexity is only rivalled by Nature itself. At the heart of them lay the Mandelbrot set, which today has achieved fame even outside the field of dynamics. This article describes where it comes from and explores its infinite intricacies.

Iteration

The Mandelbrot set is generated by what is called iteration, which means to repeat a process over and over again. In mathematics this process is most often the application of a mathematical function. For the Mandelbrot set, the functions involved are some of the simplest imaginable: they all are what is called *quadratic polynomials* and have the form $f(x) = x^2 + c$, where c is a constant number. As we go along, we will specify exactly what value c takes.

To iterate $x^2 + c$, we begin with a *seed* for the iteration. This is a number which we write as x_0 . Applying the function $x^2 + c$ to x_0 yields the new number

$$x_1 = x_0^2 + c.$$

Now, we iterate using the result of the previous computation as the input for the next. That is

$$x_2 = x_1^2 + c$$

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$$x_3 = x_2^2 + c$$

$$x_4 = x_3^2 + c$$

$$x_5 = x_4^2 + c$$

and so forth. The list of numbers x_0, x_1, x_2, \dots generated by this iteration has a name: it is called the *orbit* of x_0 under iteration of $x^2 + c$.

The theory of iterated functions is motivated by questions from real life. Modelling the growth of a population of animals is an example. The size of the population after one breeding cycle depends on how many animals there are at present, so mathematical models of population growth typically consist of a function f in a variable x , where x represents the present population size, and $f(x)$ gives the expected population size after one breeding cycle. To work out the size of the population after any number of breeding cycles you need to iterate the function. Incidentally, the functions used in a standard model of population growth are quadratic polynomials very similar to the ones we will consider here, and this is what first motivated their study.

This leads to one of the principal questions in this area of mathematics: what is the fate of typical orbits? Do they converge or diverge? Do they cycle or behave erratically? The Mandelbrot set is a geometric version of the answer to this question.

Let's begin with a few examples. Suppose we start with the constant $c = 1$. Then, if we choose the seed 0, the orbit is

$$x_0 = 0$$

$$x_1 = 0^2 + 1 = 1$$

$$x_2 = 1^2 + 1 = 2$$

$$x_3 = 2^2 + 1 = 5$$

$$x_4 = 5^2 + 1 = 26$$

$$x_5 = 26^2 + 1 = \text{BIG}$$

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$$x_6 = \text{BIGGER}$$

$$x_7 = \text{REALLY BIG},$$

and we see that points in this orbit get bigger and bigger the orbit *tends to infinity*.

As another example, choose $c = 0$. Now the orbit of the seed 0 is quite different: it remains fixed for all iterations.

$$x_0 = 0$$

$$x_1 = 0^2 + 0 = 0$$

$$x_2 = 0^2 + 0 = 0$$

$$x_3 = 0^2 + 0 = 0.$$

If we now choose $c = -1$, something else happens. For the seed 0, the orbit is

$$x_0 = 0$$

$$x_1 = 0^2 - 1 = -1$$

$$x_2 = (-1)^2 - 1 = 0$$

$$x_3 = 0^2 - 1 = -1$$

$$x_4 = (-1)^2 - 1 = 0.$$

Here we see that the orbit bounces back and forth between 0 and -1 . This is a *cycle of period 2*.

To understand the fate of orbits, it is most often easiest to proceed geometrically: a *time series plot* of the orbit often gives more information about its fate. In the plots below, we have displayed the time series for $x^2 + c$ where $c = -1.1, -1.3, -1.38$, and 1.9 . In each case we have computed the orbit of 0 and marked the points in it by dots which are connected by straight line segments. Note that the fate of the orbit changes with c . For $c =$

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-1.1 , we see that the orbit approaches a 2-cycle. For $c = -1.3$, the orbit tends to a 4-cycle. For $c = -1.38$, we see an 8-cycle. And when $c = -1.9$, there is no apparent pattern for the orbit; mathematicians use the word *chaos* for this phenomenon.

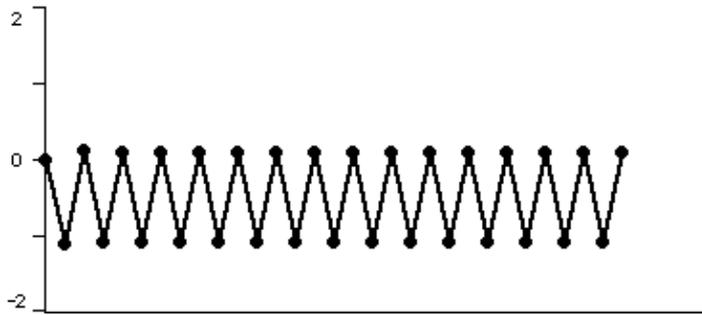


Figure 1: the orbit of 0 for iteration of $x^2 - 1.1$. The orbit approaches a 2-cycle.

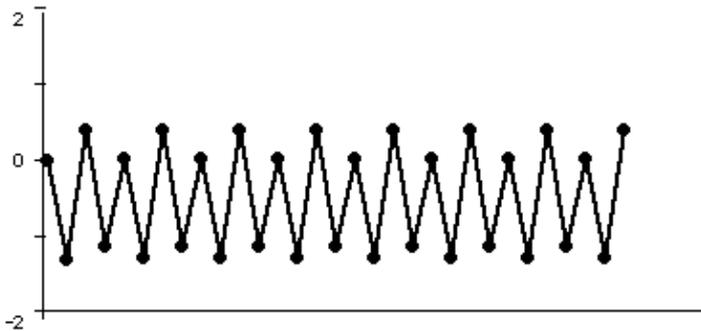


Figure 2: the orbit of 0 for iteration of $x^2 - 1.3$. The orbit approaches a 4-cycle.

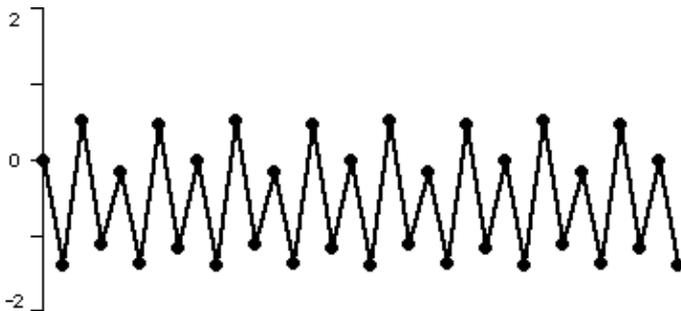
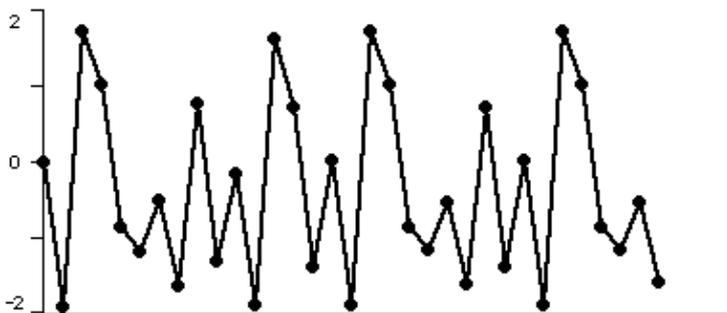


Figure 3: the orbit of 0 for iteration of $x^2 - 1.38$. The orbit approaches an 8-cycle.



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Figure 4: the orbit of 0 for iteration of $x^2 - 1.9$. There is no apparent pattern, we see chaos.

To see additional time series plots for other values of c , select a c value from the options below:

- $c = -0.65$ (Tends to a fixed point)
- $c = -1.6$ (Chaotic behaviour)
- $c = -1.75$ (Tends to period 3)
- $c = -1.8$ (Chaotic behaviour close to 3-cycle, sometimes called intermittency)
- $c = -1.85$ (Chaotic behaviour)
- $c = 0.2$ (Tends to a fixed point).

Before proceeding, let us make a seemingly obvious and uninspiring observation. Under iteration of $x^2 + c$, either the points in the orbit of 0 get larger and larger so that the orbit tends to infinity, or they do not. When the orbit does not go to infinity, it may behave in a variety of ways. It may be fixed or cyclic or behave chaotically, but the fundamental observation is that there is a *dichotomy*: sometimes the orbit goes to infinity, other times, it does not. The Mandelbrot set is a picture of precisely this dichotomy in the case where 0 is used as the seed. Thus the Mandelbrot set is a record of the fate of the orbit of 0 under iteration of $x^2 + c$: the numbers c are represented graphically and coloured a certain colour depending on the fate of the orbit of 0.

Complex numbers

How, then, is the Mandelbrot set a picture in the plane, rather than on the number line on which all the c -values we have considered lie? The answer is, instead of considering only *real* values of c , we also allow c to be a *complex number*. If you are not familiar with complex numbers, then read this [brief introduction](#) or read *Plus* article [Curious quaternions](#) for more detail.

Let's look at some examples of the iteration of $x^2 + c$ when c is a complex number: if $c=i$, then the orbit of 0 under $x^2 + i$ is given by

$$x_0 = 0$$

$$x_1 = 0^2 + i = i$$

$$x_2 = i^2 + i = -1 + i$$

$$x_3 = (-1+i)^2 + i = -i$$

$$x_4 = (-i)^2 + i = -1 + i$$

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$$x_5 = (-1+i)^2 + i = -i$$

$$x_6 = (-i)^2 + i = -1+i$$

and we see that this orbit eventually cycles with period 2. If we change c to $2i$, then the orbit behaves very differently

$$x_0 = 0$$

$$x_1 = 0^2 + 2i = 2i$$

$$x_2 = (2i)^2 + 2i = -4 + 2i$$

$$x_3 = (-4 + 2i)^2 + 2i = 12 - 14i$$

$$x_4 = (12 - 14i)^2 + 2i = -52 - 334i$$

$$x_5 = \text{BIG (meaning far away from the point 0)}$$

$$x_5 = \text{BIGGER}$$

and we see that this orbit tends to infinity in the complex plane (the numbers comprising the orbit recede further and further from the point 0, which has co-ordinates $(0,0)$). Again we make the fundamental observation that either the orbit of 0 under $x^2 + c$ tends to infinity, or it does not.

The Mandelbrot set

The Mandelbrot set puts some geometry into the fundamental observation above. Here is its precise definition:

The Mandelbrot set consists of all of those (complex) c -values for which the corresponding orbit of 0 under $x^2 + c$ does not escape to infinity.

Unveiling the Mandelbrot set

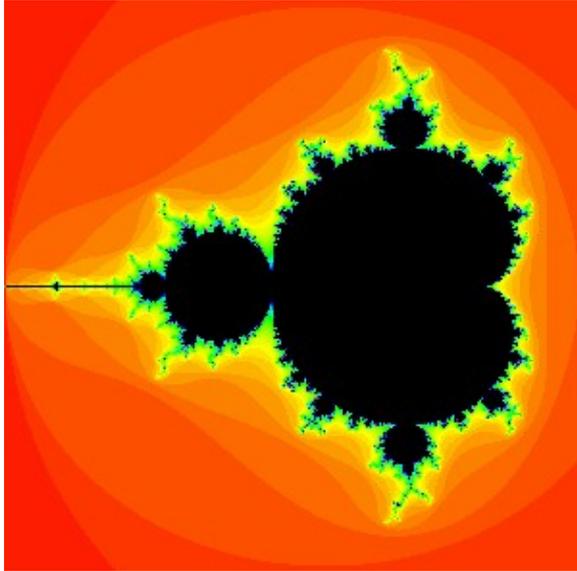


Figure 5: the black region is the Mandelbrot set pick any c -value from this black region and you will find that when you iterate x^2+c the orbit of zero does not escape to infinity. The Mandelbrot set is symmetric with respect to the x -axis in the plane, and its intersection with the x -axis occupies the interval from -2 to $1/4$. The point 0 lies within the 'main cardioid', and the point -1 lies within the 'bulb' attached to the left of the main cardioid.

From our previous calculations, we see that $c = 0, -1, -1.1, -1.3, -1.38$, and i all lie in the Mandelbrot set, whereas $c = 1$ and $c = 2i$ do not. We will represent the Mandelbrot set by the letter M . It is named after the mathematician Benoit Mandelbrot who was one of the first to study it in 1980.

At this point, a natural question is: why would anyone care about the fate of the orbit of 0 under $x^2 + c$? Why not the orbit of i ? Or $2 + 3i$? Or any other seed, for that matter? As we will see below, there is a very good reason for inquiring about the fate of the orbit of 0 ; amazingly, the orbit of 0 somehow tells us a tremendous amount about the fate of all other orbits under $x^2 + c$.

We remark at this point that our definition of the Mandelbrot set gives quite an easy way of drawing it using a computer, which is described in the *Plus* article [Computing the Mandelbrot set.](#)

Bulbs and antennas

Computer images of the Mandelbrot set, like the one above, give a hint of how incredibly intricate it is. It is known that, no matter how closely you zoom in on its boundary, it will always appear just as crinkly as it did before. The exact nature of the boundary's crinkliness is still one of the major open questions in maths. There are, however, quite a lot of things we *can* say about the Mandelbrot set, and these things reveal that its structure is by no means co-incidental: every little piece of it is loaded with mathematical meaning.

So let's have a close look at it. You can see that it consists of a main body, which looks a bit like a heart lying on its side and is therefore called the *main cardioid* of M (from the Greek *kardia*, meaning heart). Attached to the main cardioid are myriad "decorations". Closer inspection of these shows that all of them are different in shape.

Unveiling the Mandelbrot set

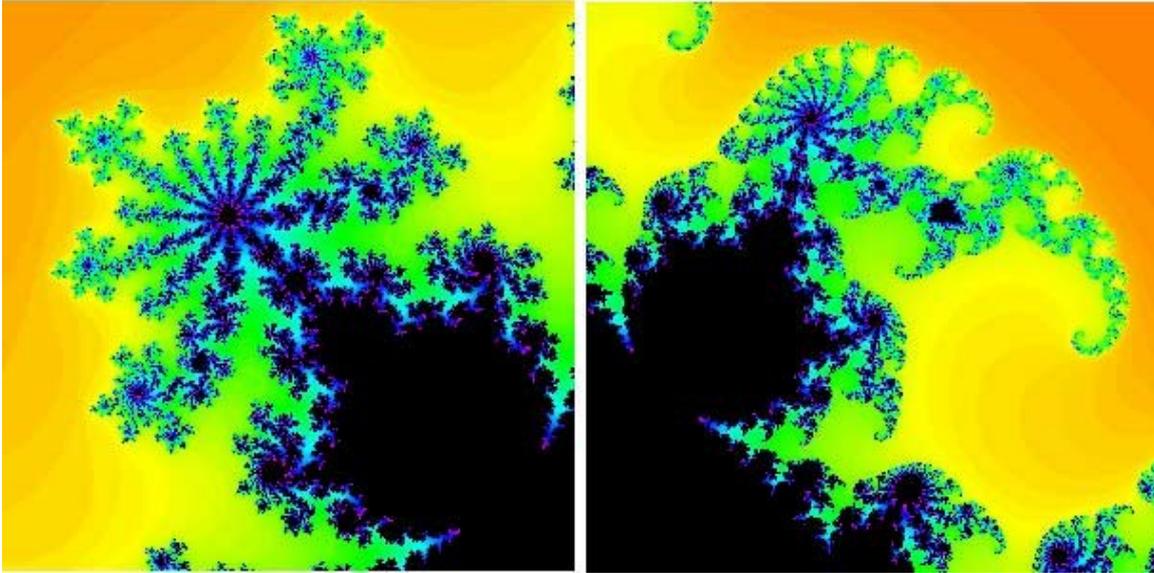
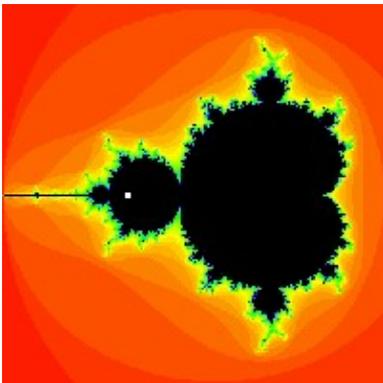


Figure 6: decorations of the Mandelbrot set in close-up. The two 'bulbs' shown here are directly attached to the main cardioid.

The decorations directly attached to the main cardioid in M are called *primary bulbs* or *primary decorations*. Any primary bulb in turn has infinitely many smaller decorations attached, as well as what appear to be antennas. In particular, as is clearly visible in figure 6, the "main antenna" attached to each decoration seems to consist of a number of spokes which varies from decoration to decoration.

There is a beautiful relationship between the number of spokes on these antennas and the dynamics of $x^2 + c$ for c lying inside the primary bulb. To see this, let's take a c value inside the big bulb that is attached to the main cardioid on the left, for example $c = -1.1$. Starting with the seed 0.0099 , you can calculate that this function has a cycle of period two:



The point $c = -1.1$ is marked in white..

$$x_0 = 0.0099$$

$$x_1 = 0.0099^2 - 1.1 = -1.0099$$

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$$x_2 = (-1.0099)^2 - 1.1 = 0.0099$$

$$x_3 = 0.0099^2 - 1.1 = -1.0099.$$

The Mandelbrot set keeps track of the orbit of 0, and if we use 0 as a seed we get:

$$\begin{aligned}x_0 &= 0, \\x_1 &= -1.1, \\x_2 &= 0.0101, \\x_3 &= -1.00989799, \\x_4 &= 0.00989595, \\x_5 &= -1.00990211, \\x_6 &= 0.009902272,\end{aligned}$$

and so forth.

The points in the orbit get closer and closer to the 2-cycle, without ever landing right on it. Mathematicians say that the orbit of zero *is attracted to* the two cycle.

This is the case not only for $c = -1.1$: for *any* c value that lies inside the same bulb as -1.1 the function $x^2 + c$ has a 2-cycle and the orbit of zero is attracted to it.

Something similar happens for *every* primary bulb: if c lies in the interior of such a primary bulb, then the orbit of 0 is attracted to a cycle of some period n . The number n is the same for any c inside this primary bulb, and it is called the *period* of the bulb. To see that this is really true, open the [Mandelbrot set iterator applet](#) on my website, which was written by James Denvir. Clicking on a point in the Mandelbrot set shown in this applet will display the orbit of zero on the left. You will see that after a few iterations this orbit cycles.

Using the applet you can verify that the periods of some of the larger primary bulbs are as shown in figure 7. Simply click on a c -value inside a bulb and count the number of points in the cycle.

Unveiling the Mandelbrot set

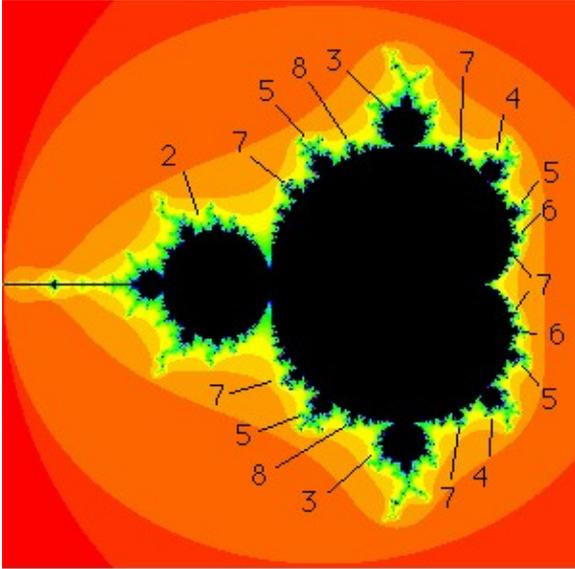


Figure 7: the periods of the bulbs

Figure 8 shows close-ups of the period 3, 4, 5 and 7 bulbs (in clockwise order starting from the top left). Now count the spokes of the largest antenna attached to each bulb (don't forget to count the spoke emanating from the primary bulb to the main junction point of the antenna.) What we are beginning to see here is a remarkable relationship: the number of spokes is precisely the period of the primary bulb!

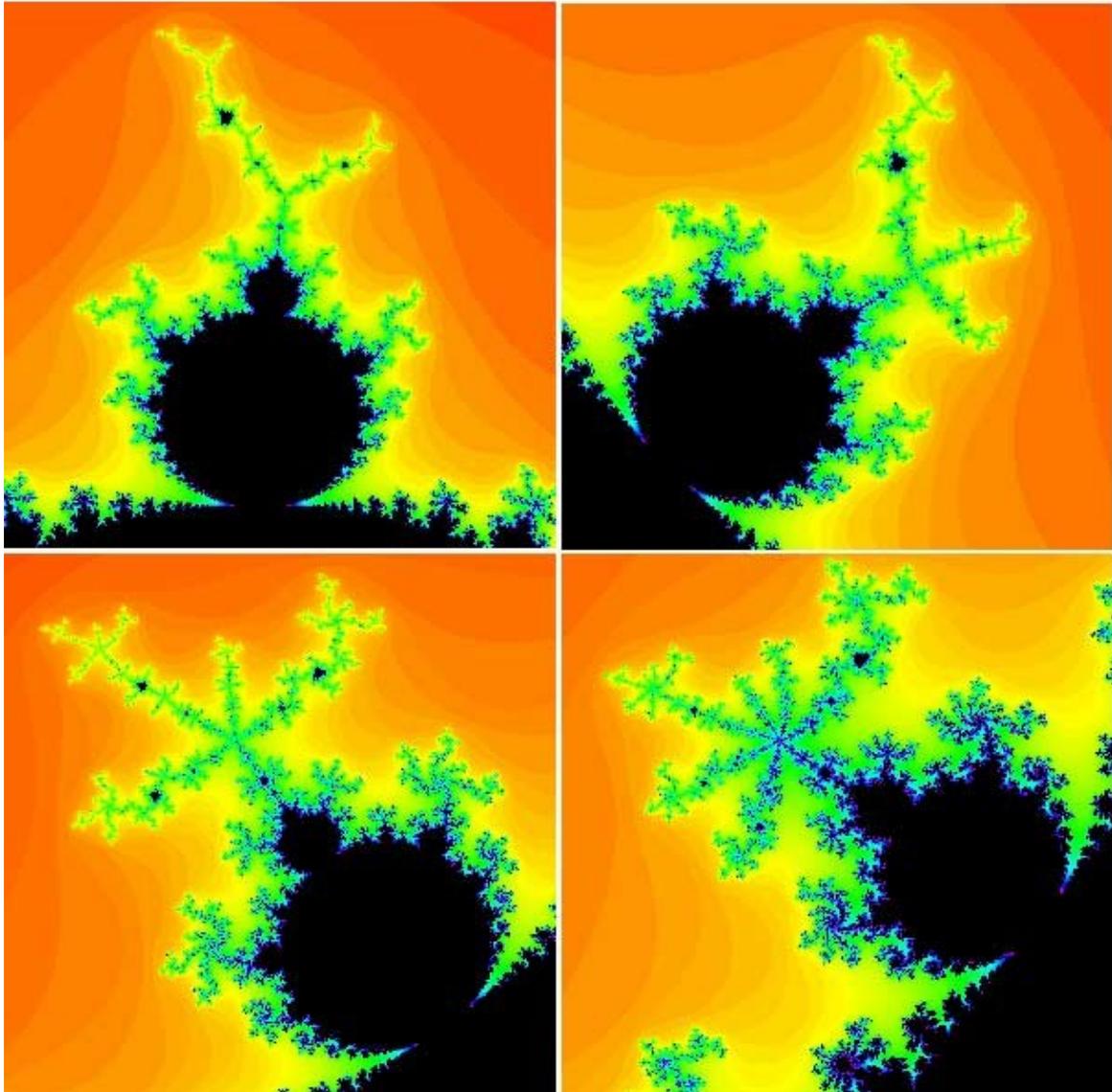


Figure 8: various bulbs and their periods. Clockwise from top left: period 3 bulb, period 4 bulb, period 5 bulb, period 7 bulb. Note that the period of the bulb is equal to the number of spokes on the main antenna.

A similar result is true for non-primary bulbs that are not directly attached to the main cardioid. In this case, the period of the bulb is a *multiple* of the number of spokes attached to its antenna.

Filled Julia sets

But the remarkable relationships don't end here. There is a second, more dynamic way to calculate the periods of these primary bulbs in M . To explain this, we have to introduce the notion of a *filled Julia set*. The filled Julia set for $x^2 + c$ is subtly different from the Mandelbrot set. For M , we calculated only the orbit of 0 for each c -value and then displayed the result. A c -value lies in M if the corresponding orbit of 0 does not escape to infinity. The Mandelbrot set is a picture in the c -plane.

For the filled Julia sets, we *fix a c -value* and then consider the fate of *all possible seeds* for that fixed value of c . Those seeds whose orbits do not escape form the filled Julia set of $x^2 + c$. Putting it formally:

Unveiling the Mandelbrot set

Fix a value of c . The filled Julia set of $x^2 + c$ is the collection of seeds whose orbit does not tend to infinity.

(Caution: There is also a notion of the *Julia set*. This term is used to describe the boundary of the filled Julia set.)

From our examples above we see that the point 0 lies in the filled Julia set of $x^2 + 0$, because it is fixed and does not go anywhere. But 0 is not in the filled Julia set of $x^2 + 1$, because it escapes to infinity.

Thus, we get a different filled Julia set for each different choice of c . We write J_c for the filled Julia set of $x^2 + c$. In figures 9 to 12 we have displayed the filled Julia sets for a variety of c -values.

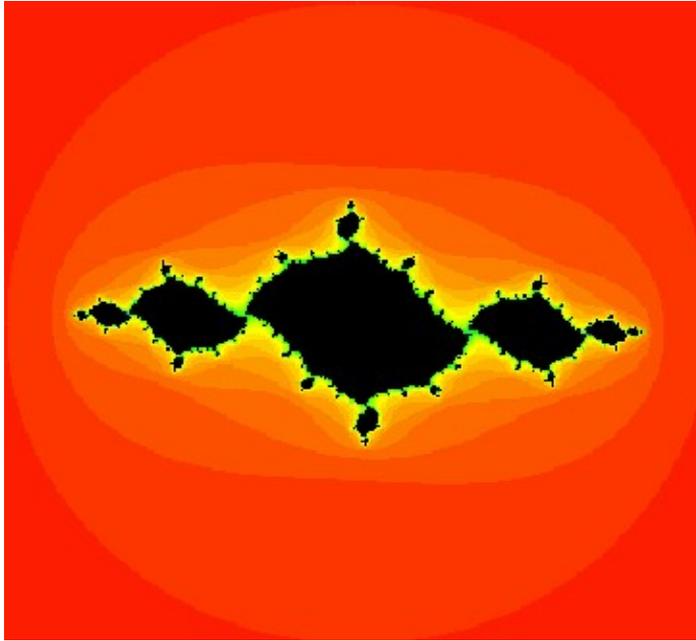


Figure 9: the filled Julia set for $c = -1.037 + 0.17i$, which lies in a period 2 bulb.

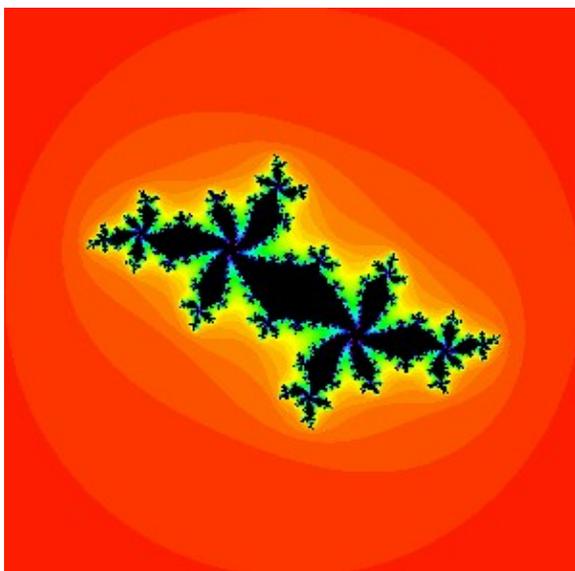


Figure 10: the filled Julia set for $c = -0.52 + 0.57i$, which lies in a period 5 bulb.

Unveiling the Mandelbrot set

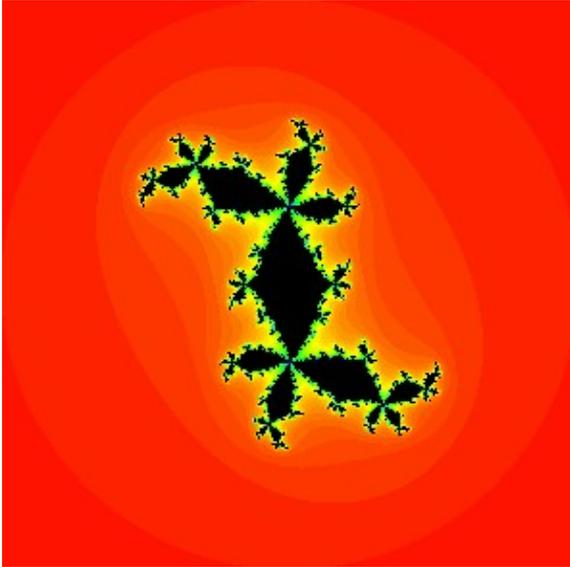


Figure 11: the filled Julia set for $c = 0.295 + 0.55i$, which lies in a period 4 bulb.

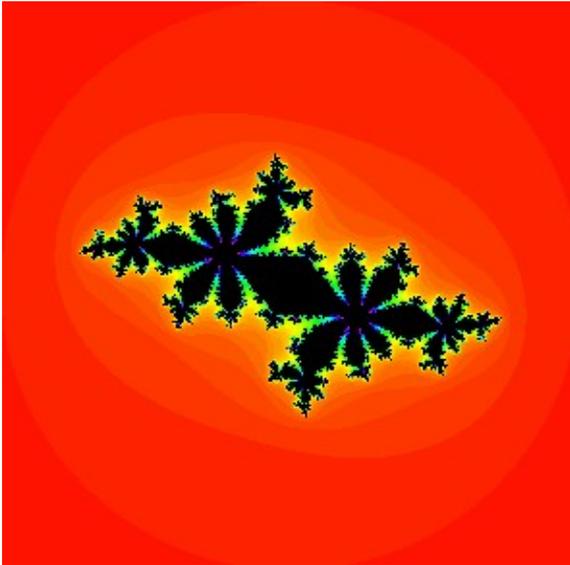


Figure 12: the filled Julia set for $c = -0.624 + 0.435i$, which lies in a period 7 bulb.

To get a feel for filled Julia sets, see if you can [show that the filled Julia set for \$c = 0\$ is a round disc](#). Most filled Julia sets, however, are impossible to work out without a computer. To see some of them open the [Mandelbrot and Julia set applet](#) on my website, which was written by [Yakov Shapiro](#). Clicking on a point in the window with the Mandelbrot set on the right specifies a c -value, and then clicking "compute" will display the corresponding filled Julia set on the left.

To summarise, here is the difference between Mandelbrot and Julia sets again:

The Mandelbrot set

- is a picture in the c -plane, also called the *parameter plane*,

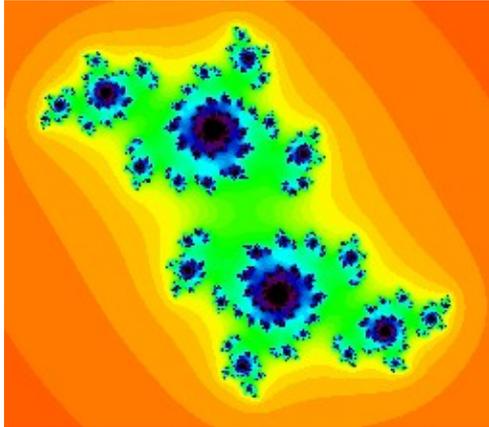
Unveiling the Mandelbrot set

- records the fate of the orbit of 0 for all possible c -values.

The filled Julia set

- is a picture in the x -plane, also called the *dynamical plane*,
- records the fate of *all* orbits for $x^2 + c$ for a *fixed* c . There is one Julia set for each value of c .

The fundamental dichotomy



This Julia set is a Cantor set. It is made up of the black points. Every such point forms a separate component, and these point components pile up on each other.

All the filled Julia sets we have displayed here have one thing in common: they are *connected sets*, in other words they consist of just one piece. But not all filled Julia sets are connected. One of the most beautiful results in all of complex dynamics dates back to 1919 and was proved independently by both Gaston Julia and Pierre Fatou. It states that either a Julia set is connected, or it consists of infinitely many pieces, each of which is a single point. These points pile up on each other creating something resembling a cloud. "Cloud sets" like these are called *Cantor sets* (see *Plus* article How big is the milky way? for more on Cantor sets).

So the *fundamental dichotomy* says that filled Julia sets for $x^2 + c$ come in one of two varieties: connected sets (one piece) or Cantor sets (infinitely many pieces). There is no in-between: there are no c -values for which J_c consists of 10 or 20 or 756 pieces.

How do we decide what shape a given J_c assumes, whether it is connected or a Cantor set? Amazingly, it is the orbit of 0 that determines this. For if the orbit of 0 tends to infinity under iteration of $x^2 + c$, then the fact is that J_c is a Cantor set. On the other hand, if the orbit of 0 does not tend to infinity, then J_c is a connected set.

A visual way to view this dichotomy is given by the Mandelbrot set. If c lies in M , then we know that the orbit of 0 does not escape to infinity under iteration of $x^2 + c$, so J_c must be connected. If c does not lie in M then J_c is a Cantor set. This dichotomy thus gives us a second interpretation of the Mandelbrot set.

The Mandelbrot set consists of all c -values for which

- J_c is connected, or, equivalently
- the orbit of 0 under $x^2 + c$ does not tend to infinity.

To see how the filled Julia sets "fall apart" into infinitely many pieces as the c -value leaves the Mandelbrot

Unveiling the Mandelbrot set

set have a look at these [movies](#) on my website.

It is amazing that the orbit of 0 "knows" the shape of the filled Julia set for $x^2 + c$. There are some deeper mathematical reasons for this, which we'll not go into here.

Back to the Mandelbrot set

But the shapes of the filled Julia set and the Mandelbrot set are connected in more ways than this. The appearance of the Julia set belonging to a c -value inside one of the decorations of M gives us a way of telling the period of that decoration, and vice versa. In figure 13 we have displayed the filled Julia set for $c = -0.12 + 0.75i$. This filled Julia set is often called *Douady's rabbit*, after Adrien Douady who is one of the pioneers in this area of maths. Note that the image looks like a "fractal rabbit." The rabbit has a main body with two ears attached. Everywhere you look you see other pairs of ears.

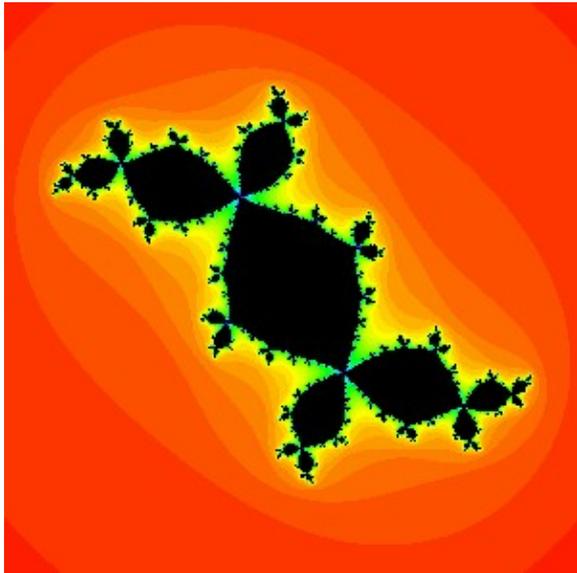


Figure 13: the fractal rabbit.

Another way to say this is that the filled Julia set contains infinitely many "junction points" at which 3 distinct black regions in J_c are attached. In figure 14 we have magnified a portion of the fractal rabbit to illustrate this.

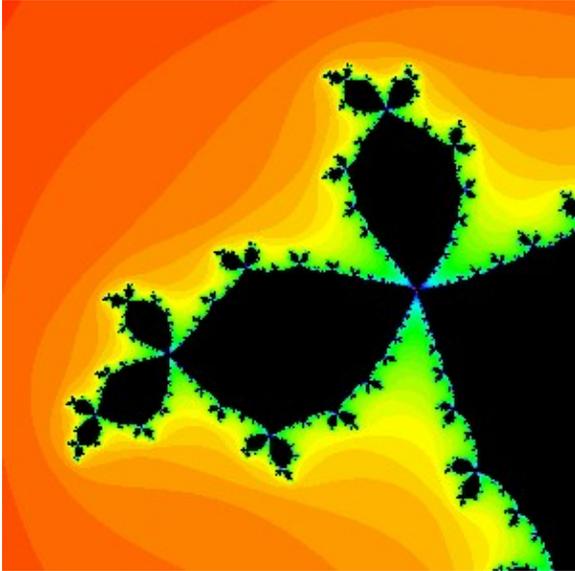


Figure 14: a magnification of the fractal rabbit.

The fact that each junction point in this filled Julia set has 3 pieces attached is no surprise to those in the know, since this c -value lies in a primary period 3 bulb in the Mandelbrot set. This is another fascinating fact about M . If you choose a c -value from one of the primary decorations in M , then, first of all, J_c must be a connected set, and secondly, J_c contains infinitely many special junction points and each of these points has exactly n regions attached to it, where n is exactly the period of the bulb. Figure 15 illustrates this for periods 4 and 5.

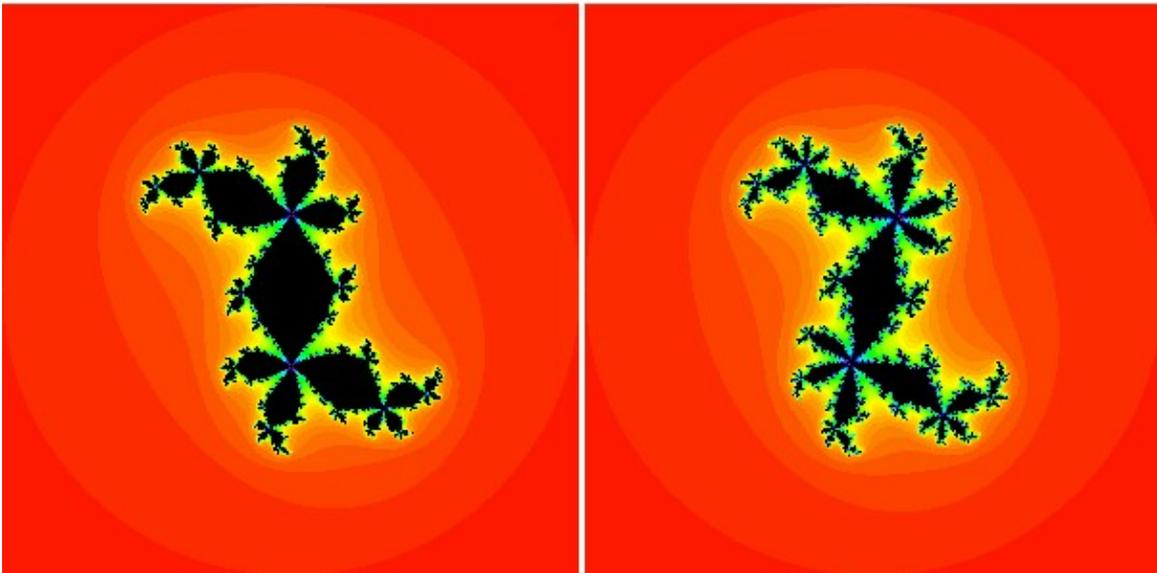


Figure 15: period 4 and period 5 filled Julia sets.

My website has a collection of [movies](#) showing how the Julia set changes and how the junction points arise as the c -value moves from the main cardioid of the Mandelbrot set into the primary bulbs.

Summary

Thus, we've seen that the Mandelbrot set possesses an extraordinary amount of structure. We can use the geometry of M to understand the dynamics of $x^2 + c$. Or we can take dynamical information and use it to understand the shape of M . This interplay between dynamics and geometry is on the one hand fascinating and, on the other, still not completely understood. Much of this interplay has been catalogued in recent years by mathematicians such as Adrien Douady, John H Hubbard, Jean-Christophe Yoccoz, Curt McMullen, and others, but much more remains to be discovered. If you enjoyed this article, then maybe you can continue studying the Mandelbrot set and become one of the people making the discoveries.

About the author



Robert Devaney. [Find out](#) where this picture was taken.

[Robert Devaney](#) is currently Professor of Mathematics at Boston University. His main area of research is dynamical systems, primarily complex analytic dynamics, but also including more general ideas about chaotic dynamical systems. He is the author of over one hundred research and pedagogical papers in the field of dynamical systems. He is also the (co)-author or editor of thirteen books in this area of mathematics, including a series of four books collectively called [A Tool Kit of Dynamics Activities](#), all aimed at high school students and teachers. He has also been the "Chaos Consultant" for several theaters' presentations of Tom Stoppard's play [Arcadia](#).

Professor Devaney has delivered over 1,300 invited lectures on dynamical systems and related topics in all 50 states in the US and in over 30 countries on six continents worldwide. He only needs Antarctica to complete his goal of speaking on all continents so if you teach at South Pole State and run some kind of seminar, give him a call!

Professor Devaney's [website](#) contains a number of interesting applets, articles and interactive papers on dynamical systems. Have a look.



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